Modelling Dependence in High Dimensions with Factor Copulas*

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Abstract

This paper presents new models for the dependence structure, or copula, of economic variables based on a factor structure. The proposed models are particularly attractive for high dimensional applications, involving fifty or more variables. This class of models generally lacks a closed-form density, but analytical results for the implied tail dependence can be obtained using extreme value theory, and estimation via a simulation-based method using rank statistics is simple and fast. We study the finite-sample properties of the estimation method for applications involving up to 100 variables, and apply the model to daily returns on all 100 constituents of the S&P 100 index. We find significant evidence of tail dependence, heterogeneous dependence, and asymmetric dependence, with dependence being stronger in crashes than in booms. We also show that the proposed factor copula model provides superior estimates of some measures of systemic risk.

Keywords: correlation, dependence, copulas, tail dependence, systemic risk.

J.E.L. codes: C31, C32, C51.

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1 Introduction

One of the many surprises from the financial crisis of late 2007 to 2008 was the extent to which assets that had previously behaved mostly independently suddenly moved together. This was particularly prominent in the financial sector, where poor models of the dependence between certain asset returns (such as those on housing, or those related to mortgage defaults) are thought to be one of the causes of the collapse of the market for CDOs and related securities, see Coval *et al.* (2009) and Zimmer (2012) for example. Many models that were being used to capture the dependence between a large number of financial assets were revealed as being inadequate during the crisis. However, one of the difficulties in analyzing risks across many variables is the relative paucity of econometric models suitable for the task. Correlation-based models, while useful when risk can be summarized using the second moment, are often built on an assumption of multivariate Gaussianity, and face the risk of neglecting dependence between the variables in the tails, i.e., neglecting the possibility that large crashes may be correlated across assets.

This paper makes two primary contributions. First, we present new models for the dependence structure, or copula, of economic variables. The models are based on a simple factor structure for the copula and are particularly attractive for high dimensional applications, involving fifty or more variables.[1] These copula models may be combined with existing models for univariate distributions to construct flexible, tractable joint distributions for large collections of variables. The proposed copula models permit the researcher to determine the degree of flexibility based on the number of variables and the amount of data available. For example, by allowing for a fat-tailed common factor the model captures the possibility of correlated crashes, and by allowing the common factor to be asymmetrically distributed the model allows for the possibility that the dependence between the variables is stronger during downturns than during upturns. By allowing for multiple common factors, it is possible to capture heterogeneous pair-wise dependence within the overall multivariate copula. High dimension economic applications will often require some strong simplifying assumptions in order to keep the model tractable, and an important feature of the class of proposed models is that such assumptions can be made in an easily understandable manner, and can be tested and relaxed if needed.

[1] For recent work on high dimensional covariance matrix estimation, see Engle *et al.* (2008), Fan *et al.* (2008), Engle and Kelly (2012), Fan *et al.* (2012) and Hautsch *et al.* (2012).

Factor copulas do not generally have a closed-form density, but certain properties can nevertheless be obtained analytically. Using extreme value theory we obtain theoretical results on the tail dependence properties for general, multi-factor copulas, and for the specific parametric class of factor copulas that we use in our empirical work.

The second contribution of this paper is a study of the dependence structure of all 100 constituent firms of the Standard and Poor's 100 index, using daily data over the period 2008-2010. This is one of the highest dimension applications of copula theory in the econometrics literature. We find significant evidence in favor of a fat-tailed common factor for these stocks (indicative of non-zero tail dependence), implying that the Normal, or Gaussian, copula is not suitable for these assets. Moreover, we find significant evidence that the common factor is asymmetrically distributed, with crashes being more highly correlated than booms. Our empirical results suggest that risk management decisions made using the Normal copula may be based on too benign a view of these assets, and derivative securities based on baskets of these assets, or related securities such as CDOs, may be mispriced if based on a Normal copula. The fact that large negative shocks may originate from a fat-tailed common factor, and thus affect all stocks at once, makes the diversification benefits of investing in these stocks lower than under Normality. In an application to estimating systemic risk, we show that our factor copula model provides superior estimates of two measures of systemic risk.

An additional contribution of this paper is a detailed simulation study of the properties of the estimation method for the class of factor copulas we propose. This class does not generally have a closed-form copula likelihood, and we use the SMM estimator proposed in Oh and Patton (2011). We consider problems of dimension 3, 10 and 100, and confirm that the estimator and associated asymptotic distribution theory have satisfactory finite-sample properties.

Certain types of factor copulas have already appeared in the literature. The models we consider are extensions of Hull and White (2004), in that we retain a simple linear, additive factor structure, but allow for the variables in the structure to have flexibly specified distributions. Other variations on factor copulas are presented in Andersen and Sidenius (2004) and van der Voort (2005), who consider certain non-linear factor structures, and in McNeil *et al.* (2005), who present factor copulas for modelling times-to-default. With the exception of McNeil *et al.* (2005), the papers to date have not considered estimation of the unknown parameters of these copulas, instead examining calibration and pricing using these copulas. Our formal analysis of the estimation of high dimension copulas via a SMM-type procedure is new to the literature, as is our application of this class of models to a large collection of asset returns.

Some methods for modelling high dimension copulas have previously been proposed in the literature, though few consider dimensions greater than twenty.[2] The Normal copula, see Li (2000) amongst many others, is simple to implement and to understand, but imposes the strong assumption of zero tail dependence, and symmetric dependence between booms and crashes. The (Student's) t copula, and variants of it, are discussed in Demarta and McNeil (2005). An attractive extension of the t copula, the "grouped t" copula, is proposed in Daul *et al.* (2003), who show that this copula can be used in applications of up to 100 variables. This copula allows for heterogeneous tail dependence between pairs of variables, but imposes that upper and lower tail dependence are equal (a finding we strongly reject for equity returns). Smith, *et al.* (2012) extract the copula implied by a multivariate skew t distribution, and Christoffersen *et al.* (2011) combine a skew t copula with a DCC model for conditional correlations in their study of 33 developed and emerging equity market indices. Archimedean copulas such as the Clayton or Gumbel allow for tail dependence and particular forms of asymmetry, but usually have only a one or two parameters to characterize the dependence between all variables, and are thus quite restrictive when the number of variables is large. Multivariate "vine" copulas are constructed by sequentially applying bivariate copulas to build up a higher dimension copula, see Aas *et al.* (2009), Heinen and Valdesogo (2009) and Min and Czado (2010) for example, however vine copulas are almost invariably based on an assumption that is hard to interpret and to test, see Acar *et al.* (2012) for a critique. In our empirical application we compare our proposed factor models with several alternative existing models, and show that our model outperforms them all in terms of goodness-of-fit and in an application to measuring systemic risk.

The remainder of the paper is structured as follows. Section 2 presents the class of factor copulas, derives their limiting tail properties, and considers some extensions. Section 3 considers estimation via a simulation-based method and presents a simulation study of this method. Section 4 presents an empirical study of daily returns on individual constituents of the S&P 100 equity index over the period 2008-2010. Appendix A contains all proofs, and Appendix B contains a discussion of the dependence measures used in estimation.

[2] For general reviews of copulas in economics and finance see Cherubini, *et al.* (2004) and Patton (2012).

2 Factor copulas

For simplicity of exposition we focus on unconditional distributions in this section, and discuss the extension to conditional distributions in the next section. Consider a vector of N variables, $\mathbf{Y}$, with some joint distribution $\mathbf{F}$, marginal distributions F_i, and copula $\mathbf{C}$:

$$[Y_1, ..., Y_N]' \equiv \mathbf{Y} \sim \mathbf{F} = \mathbf{C}\left(F_1, ..., F_N\right) \tag{1}$$

The copula completely describes the dependence between the variables $Y_1, ..., Y_N$. We will use existing models to estimate the marginal distributions F_i (which may be parametric, semiparametric or nonparametric), and focus on constructing useful new models for the dependence between these variables, $\mathbf{C}$.[3] Decomposing the joint distribution in this way has two important advantages over considering the joint distribution $\mathbf{F}$ directly: First, it facilitates multi-stage estimation, which is particularly useful in high dimension applications, where the sparseness of the data and the potential proliferation of parameters can cause problems. Second, it allows the researcher to draw on the large literature on models for univariate distributions, leaving "only" the task of constructing a model for the copula, which is a simpler problem.

2.1 Description of a simple factor copula model

The class of copulas we consider are those that can be generated by the following simple factor structure, based on a set of $N+1$ latent variables:

$$\begin{aligned} X_i &= Z + \varepsilon_i, \;\; i = 1, 2, ..., N \\ Z &\sim F_z\left(\boldsymbol{\theta}\right), \;\; \varepsilon_i \sim iid\ F_\varepsilon\left(\boldsymbol{\theta}\right), \;\; Z \perp\!\!\!\perp \varepsilon_i \;\forall\; i \\ [X_1, ..., X_N]' &\equiv \mathbf{X} \sim \mathbf{F}_x = \mathbf{C}\left(G_1\left(\boldsymbol{\theta}\right), ..., G_N\left(\boldsymbol{\theta}\right); \boldsymbol{\theta}\right) \end{aligned} \tag{2}$$

The copula of the latent variables $\mathbf{X}$, $\mathbf{C}\left(\boldsymbol{\theta}\right)$, is used as the model for the copula of the observable variables $\mathbf{Y}$.[4] An important point about the above construction is that the marginal distributions

[3] Although we treat estimation of the marginal distributions as separate from copula estimation, the inference methods we consider *do* take estimation error from the marginal distributions into account.

[4] This method for constructing a copula model resembles the use of mixture models, e.g. the Normal-inverse Gaussian or generalized hyperbolic distributions, where the distribution of interest is obtained by considering a function of a collection of latent variables, see Barndorff-Nielsen (1978, 1997), Barndorff-Nielsen and Shephard (2009), McNeil, *et al.* (2005).

of X_i may be different from those of the original variables Y_i, so $F_i \neq G_i$ in general. We use the structure for the vector $\mathbf{X}$ *only* for its copula, and completely discard the resulting marginal distributions. By doing so, we use $\mathbf{C}(\boldsymbol{\theta})$ from equation (2) to construct a model for the *copula* of $\mathbf{Y}$, and leave the marginal distributions F_i to be specified and estimated in a separate step.

The copula implied by the above structure is generally not known in closed form. The leading case where it *is* known is when F_z and F_ε are both Gaussian distributions, in which case the variable $\mathbf{X}$ is multivariate Gaussian, implying a Gaussian copula, and with an equicorrelation dependence structure (with correlation between any pair of variables equal to $\sigma_z^2/\left(\sigma_z^2+\sigma_\varepsilon^2\right)$). For other choices of F_z and F_ε the joint distribution of $\mathbf{X}$, and more importantly the copula of $\mathbf{X}$, is generally not known in closed form. It is clear from the structure above that the copula will exhibit "equidependence", in that each pair of variables will have the same bivariate copula as any other pair. (This property is known as "exchangeability" in the copula literature.) A similar assumption for correlations is made in Engle and Kelly (2012).

It is simple to simulate from F_z and F_ε for many classes of distributions, and from simulated data we can extract properties of the copula, such as rank correlation, Kendall's tau, and quantile dependence. These simulated rank dependence measures can be used in simulated method of moments (SMM) type estimation of the unknown parameters, which is described in Section 3 below.

2.2 A multi-factor copula model

The structure of the model in equation (2) immediately suggests two directions for extensions. The first is to allow for weights on the common factor that differ across variables. That is, let

$$\begin{aligned} X_i &= \beta_i Z + \varepsilon_i, \;\; i = 1,2,...,N \\ Z &\sim F_z, \;\; \varepsilon_i \sim iid\ F_\varepsilon, \;\; Z \perp\!\!\!\perp \varepsilon_i \ \forall\ i \end{aligned} \quad (3)$$

with the rest of the model left unchanged. In this "single factor, flexible weights" factor copula, the implied copula is no longer equidependent: a given pair of variables may have weaker or stronger dependence than some other pair. This extension introduces $N-1$ additional parameters to this model, increasing its flexibility to model heterogeneous pairs of variables, at the cost of a more difficult estimation problem. An intermediate model may be considered, in which sub-sets of variables are assumed to have the same weight on the common factor, which may be reasonable for financial applications with variables grouped ex ante using industry classifications, for example. Such an assumption leads to a "block equidependence" copula, and we will consider this structure in our empirical application.

A second extension to consider is a multi-factor version of the model, where the dependence is assumed to come from a K-factor model:

$$\begin{aligned} X_i &= \sum_{k=1}^{K} \beta_{ik} Z_k + \varepsilon_i \\ \varepsilon_i &\sim iid\ F_\varepsilon,\ \ Z_k \perp\!\!\!\perp \varepsilon_i\ \forall\ i,k \\ [Z_1, ..., Z_K]' &\equiv \mathbf{Z} \sim \mathbf{F}_z = \mathbf{C}_{indep}\left(F_{z_1}, ..., F_{z_K}\right) \end{aligned} \tag{4}$$

In the most general case one could allow $\mathbf{Z}$ to have a general copula $\mathbf{C}_Z$ that allows dependence between the common factors, however an empirically useful simplification of this model is to impose that the common factors are independent, and thus remove the need to specify and estimate $\mathbf{C}_Z$. A further simplification of this factor model may be to assume that each common factor has a loading equal to one or zero, with the weights specified in advance by grouping variables, for example by grouping stocks by industry.

The above model can be interpreted as a special case of the "conditional independence structure" of McNeil, *et al.* (2005), which is used to describe a set of variables that are independent conditional on some smaller set of variables, $\mathbf{X}$ and $\mathbf{Z}$ in our notation.[5] McNeil, *et al.* (2005) describe using such a structure to generate some factor copulas to model times until default.

2.3 Tail dependence properties of factor copulas

Using results from extreme value theory, it is possible to obtain analytically results on the tail dependence implied by a factor copula model despite the fact that we do not have a closed-form expression for the copula. These results are relatively easy to obtain, given the simple linear structure generating the factor copula. Recall the definition of tail dependence for two variables

[5] The variables $\mathbf{Z}$ are sometimes known as the "frailty", in the survival analysis and credit default literature, see Duffie, *et al.* (2009) for example.

$X_i,\ X_j$ with marginal distributions G_i, G_j :

$$\tau_{ij}^L \equiv \lim_{q\to 0} \frac{\Pr\left[X_i \le G_i^{-1}(q), X_j \le G_j^{-1}(q)\right]}{q} \qquad (5)$$
$$\tau_{ij}^U \equiv \lim_{q\to 1} \frac{\Pr\left[X_i > G_i^{-1}(q), X_j > G_j^{-1}(q)\right]}{1-q}$$

That is, lower tail dependence measures the probability of both variables lying below their q quantile, for q limiting to zero, scaled by the probability of one of these variables lying below their q quantile. Upper tail dependence is defined analogously. In Proposition 1 below we present results for a general single factor copula model:

Proposition 1 (Tail dependence for a factor copula) *Consider the factor copula generated by equation (3). If F_z and F_ε have regularly varying tails with a common tail index $\alpha > 0$, i.e.*

$$\Pr[Z > s] = A_z^U s^{-\alpha} \quad \textit{and} \quad \Pr[\varepsilon_i > s] = A_\varepsilon^U s^{-\alpha}, \quad \textit{as } s \to \infty \qquad (6)$$
$$\Pr[Z < -s] = A_z^L s^{-\alpha} \quad \textit{and} \quad \Pr[\varepsilon_i < -s] = A_\varepsilon^L s^{-\alpha} \quad \textit{as } s \to \infty$$

where $A_Z^L,\ A_Z^U,\ A_\varepsilon^L$ and A_ε^U are positive constants, then (a) if $\beta_j \ge \beta_i > 0$ the lower and upper tail dependence coefficients are:

$$\tau_{ij}^L = \frac{\beta_i^\alpha A_z^L}{\beta_i^\alpha A_z^L + A_\varepsilon^L}, \quad \tau_{ij}^U = \frac{\beta_i^\alpha A_z^U}{\beta_i^\alpha A_z^U + A_\varepsilon^U} \qquad (7)$$

(b) if $\beta_j \le \beta_i < 0$ the lower and upper tail dependence coefficients are:

$$\tau_{ij}^L = \frac{|\beta_i|^\alpha A_z^U}{|\beta_i|^\alpha A_z^U + A_\varepsilon^L}, \quad \tau_{ij}^U = \frac{|\beta_i|^\alpha A_z^L}{|\beta_i|^\alpha A_z^L + A_\varepsilon^U} \qquad (8)$$

(c) if $\beta_i\beta_j = 0$ or (d) if $\beta_i\beta_j < 0$, the lower and upper tail dependence coefficients are zero.

All proofs are presented in Appendix A. This proposition shows that when the coefficients on the common factor have the same sign, and the common factor and idiosyncratic variables have the same tail index, the factor copula generates upper and lower tail dependence. If either Z or ε is asymmetrically distributed, then the upper and lower tail dependence coefficients can differ, which provides this model with the ability to capture differences in the probabilities of joint crashes and joint booms. When either of the coefficients on the common factor are zero, or if they have differing signs, then it is simple to show that the upper and lower tail dependence coefficients are both zero.

The above proposition considers the case that the common factor and idiosyncratic variables have the same tail index; when these indices differ we obtain a boundary result: if the tail index of Z is strictly greater than that of ε and $\beta_i\beta_j > 0$ then tail dependence is one, while if the tail index of Z is strictly less than that of ε then tail dependence is zero.

In our simulation study and empirical work below, we will focus on the skew t distribution of Hansen (1994) as a model for the common factor and the standardized t distribution for the idiosyncratic shocks. Proposition 2 below presents the analytical tail dependence coefficients for a factor copula based on these distributions.

Proposition 2 (Tail dependence for a skew t-t factor copula) *Consider the factor copula generated by equation (3). If $F_z = Skew\ t\,(\nu,\lambda)$ and $F_\varepsilon = t\,(\nu)$, then the tail indices of Z and ε_i equal ν, and the constants A_z^L, A_z^U, A_ε^L and A_ε^U from Proposition 1 equal:*

$$\begin{aligned} A_z^L &= \frac{bc}{\nu}\left(\frac{b^2}{(\nu-2)(1-\lambda)^2}\right)^{-(\nu+1)/2}, \quad A_z^U = \frac{bc}{\nu}\left(\frac{b^2}{(\nu-2)(1+\lambda)^2}\right)^{-(\nu+1)/2} \\ A_\varepsilon^L &= A_\varepsilon^U = \frac{c}{\nu}\left(\frac{1}{\nu-2}\right)^{-(\nu+1)/2} \end{aligned} \tag{9}$$

where $a = 4\lambda c\,(\nu-2)/(\nu-1)$, $b = \sqrt{1+3\lambda^2-a^2}$, $c = \Gamma\left(\frac{\nu+1}{2}\right)/\left(\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\,(\nu-2)}\right)$. *Given Proposition 1 and the expressions for A_z^L, A_z^U, A_ε^L and A_ε^U above, we then obtain the tail dependence coefficients for this copula.*

In the next proposition we generalize Proposition 1 to allow for a multi-factor model, which will prove useful in our empirical application in Section 4.

Proposition 3 (Tail dependence for a multi-factor copula) *Consider the factor copula generated by equation (4). Assume F_ε, $F_{z_1}, ..., F_{z_K}$ have regularly varying tails with a common tail index $\alpha > 0$, and upper and lower tail coefficients $A_\varepsilon^U, A_1^U, .., A_K^U$ and $A_\varepsilon^L, A_1^L, .., A_K^L$. Then if $\beta_{ik} \geq 0$ $\forall$ i,k, the lower and upper tail dependence coefficients are:*

$$\begin{aligned} \tau_{ij}^L &= \frac{\sum_{k=1}^K \mathbf{1}\left\{\beta_{ik}\beta_{jk} > 0\right\} A_k^L \beta_{ik}^\alpha \delta_{L,ijk}^\alpha}{A_\varepsilon^L + \sum_{k=1}^K A_k^L \beta_{ik}^\alpha} \\ \tau_{ij}^U &= \frac{\sum_{k=1}^K \mathbf{1}\left\{\beta_{ik}\beta_{jk} > 0\right\} A_k^U \beta_{ik}^\alpha \delta_{U,ijk}^\alpha}{A_\varepsilon^U + \sum_{k=1}^K A_k^U \beta_{ik}^\alpha} \end{aligned} \tag{10}$$

where

$$\delta_{L,ijk}^{-1} \equiv \begin{cases} \max\left\{1, \gamma_{L,ij}\beta_{ik}/\beta_{jk}\right\}, & \text{if } \beta_{ik}\beta_{jk} > 0 \\ 1, & \text{if } \beta_{ik}\beta_{jk} = 0 \end{cases} \tag{11}$$

$$\delta_{U,ijk}^{-1} \equiv \begin{cases} \max\left\{1, \gamma_{U,ij}\beta_{ik}/\beta_{jk}\right\}, & \text{if } \beta_{ik}\beta_{jk} > 0 \\ 1, & \text{if } \beta_{ik}\beta_{jk} = 0 \end{cases}$$

$$\gamma_{L,ij} \equiv \left(\frac{A_\varepsilon^L + \sum_{k=1}^K A_k^L \beta_{jk}^\alpha}{A_\varepsilon^L + \sum_{k=1}^K A_k^L \beta_{ik}^\alpha}\right)^{1/\alpha}, \quad \gamma_{U,ij} \equiv \left(\frac{A_\varepsilon^U + \sum_{k=1}^K A_k^U \beta_{jk}^\alpha}{A_\varepsilon^U + \sum_{k=1}^K A_k^U \beta_{ik}^\alpha}\right)^{1/\alpha} \tag{12}$$

The extensions to consider the case that some have opposite signs to the others can be accommodated using the same methods as in the proof of Proposition 1. In the one-factor copula model the variables $\delta_{L,ijk}$ and $\delta_{U,ijk}$ can be obtained directly and are determined by $\min\left\{\beta_i, \beta_j\right\}$; in the multi-factor copula model these variables can be determined using equation (11) above, but do not generally have a simple expression.

2.4 Illustration of some factor copulas

To illustrate the flexibility of this simple class of copulas, Figure 1 presents 1000 random draws from bivariate distributions constructed using four different factor copulas. In all cases the marginal distributions, F_i, are set to $N(0,1)$, and the variance of the latent variables in the factor copula are set to $\sigma_z^2 = \sigma_\varepsilon^2 = 1$, so that the common factor (Z) accounts for one-half of the variance of each X_i. The first copula is generated from a factor structure with $F_z = F_\varepsilon = N(0,1)$, implying that the copula is Normal. The second sets $F_z = F_\varepsilon = t(4)$, generating a symmetric copula with positive tail dependence. The third copula sets $F_\varepsilon = N(0,1)$ and $F_z = skew\ t(\infty, -0.25)$, corresponding to a skewed Normal distribution. This copula exhibits asymmetric dependence, with crashes being more correlated than booms, but zero tail dependence. The fourth copula sets $F_\varepsilon = t(4)$ and $F_z = skew\ t(4, -0.25)$, which generates asymmetric dependence and positive tail dependence.

Figure 1 shows that when the distributions in the factor structure are Normal or skewed Normal, tail events tend to be uncorrelated across the two variables. When the degrees of freedom is set to 4, on the other hand, we observe several draws in the joint upper and lower tails. When the skewness parameter is negative, as in the lower two panels of Figure 1, we observe stronger clustering of observations in the joint negative quadrant compared with the joint positive quadrant.

An alternative way to illustrate the differences in the dependence implied by these four models is to use a measure known as "quantile dependence". This measure captures the probability of observing a draw in the q-tail of one variable given that such an observation has been observed for the other variable. It is defined as:

$$\tau_q \equiv \begin{cases} \frac{1}{q}\Pr\left[U_1 \leq q, U_2 \leq q\right], & q \in (0, 0.5] \\ \frac{1}{1-q}\Pr\left[U_1 > q, U_2 > q\right], & q \in (0.5, 1) \end{cases} \tag{13}$$

where $U_i \equiv G_i(X_i) \sim Unif(0,1)$ are the probability integral transforms of the simulated X_i variables. As $q \to 0$ ($q \to 1$) this measure converges to lower (upper) tail dependence, and for values of q "near" zero or one we obtain an estimate of the dependence "near" the joint tails.

Figure 2 presents the quantile dependence functions for these four copulas.[6] For the symmetric copulas (Normal, and t-t factor copula) this function is symmetric about $q = 0.5$, while for the others it is not. The two copulas with a fat-tailed common factor exhibit quantile dependence that increases near the tails: in those cases an extreme observation is more likely to have come from the fat-tailed common factor (Z) than from the thin-tailed idiosyncratic variable (ε_i), and thus an extreme value for one variable makes an extreme value for the other variable more likely. Figure 2 also presents the theoretical tail dependence for each of these copulas based on Proposition 2 above using a symbol at $q = 0$ (lower tail dependence) and $q = 1$ (upper tail dependence). The *skew* $t(4)$-$t(4)$ factor copula illustrates the flexibility of this simple class of models, generating weak upper quantile dependence but strong lower quantile dependence, a feature that may be useful when modelling asset returns.

Figure 3 illustrates the differences between these copulas using a truly multivariate approach: Conditional on observing k out of 100 stocks crashing, we present the expected number, or proportion, of the remaining $(100 - k)$ stocks that will crash, a measure based on Geluk, *et al.* (2007):

$$\begin{aligned} \pi^q(j) &\equiv \frac{\kappa^q(j)}{N-j} \\ \text{where} \quad \kappa^q(j) &= E\left[N_q^* | N_q^* \geq j\right] - j \\ N_q^* &\equiv \sum\nolimits_{i=1}^{N} \mathbf{1}\left\{U_i \leq q\right\} \end{aligned} \tag{14}$$

For this illustration we define a "crash" as a realization in the lower 1/66 tail, corresponding to a once-in-a-quarter event for daily asset returns. The upper panel shows that as we condition on

[6] For the Normal copula the quantile dependence function is known in closed form; for the remaining copula models we use 50,000 simulations obtain these functions.

more variables crashing, the expected number of other variables that will crash, $\kappa^q(j)$, initially increases, and peaks at around $j = 30$. At that point, a *skew* $t(4)$-$t(4)$ factor copula predicts that around another 38 variables will crash, while under the Normal copula we expect only around 12 more variables to crash. As we condition on even more variables crashing the plot converges to inevitably zero, since conditioning on having observed more crashes, there are fewer variables left to crash. The lower panel of Figure 3 shows that the expected *proportion* of remaining stocks that will crash, $\pi^q(j)$, generally increases all the way to $j = 99$.[7] For comparison, this figure also plots the results for a positively skewed *skew t* factor copula, where booms are more correlated than crashes. This copula also exhibits tail dependence, and so the expected proportion of other stocks that will crash is higher than under Normality, but the positive skew means that crashes are less correlated than booms, and so the expected proportion is less than when the common factor is negatively skewed. This figure illustrates some of the features of dependence that are unique to high dimension applications, and further motivates our proposal for a class of flexible, parsimonious models for such applications.

[INSERT FIGURES 1, 2 AND 3 ABOUT HERE]

2.5 Non-linear factor copula models

We can generalize the above linear, additive factor structure to consider more general factor structures. For example, consider the following general one-factor structure:

$$\begin{aligned} X_i &= h(Z, \varepsilon_i), \ \ i = 1, 2, ..., N \\ Z &\sim F_z, \ \ \varepsilon_i \sim iid\ F_\varepsilon, \ \ Z \perp\!\!\!\perp \varepsilon_i \ \forall\ i \\ [X_1, ..., X_N]' &\equiv \mathbf{X} \sim \mathbf{F}_x = \mathbf{C}(G_1, ..., G_N) \end{aligned} \tag{15}$$

for some function $h : \mathbb{R}^2 \to \mathbb{R}$. Writing the factor model in this general form reveals that this structure nests a variety of well-known copulas in the literature. Some examples of copula models

[7] For the Normal copula this is not the case, however this is perhaps due to simulation error: even with the 10 million simulations used to obtain this figure, joint 1/66 tail crashes are so rare under a Normal copula that there is a fair degree of simulation error in this plot for $j \geq 80$.

that fit in this framework are summarized in the table below:

Copula	$h(Z,\varepsilon)$	F_Z	F_ε
Normal	$Z+\varepsilon$	$N\left(0,\sigma_z^2\right)$	$N\left(0,\sigma_\varepsilon^2\right)$
Student's t	$Z^{1/2}\varepsilon$	$Ig\left(\nu/2,\nu/2\right)$	$N\left(0,\sigma_\varepsilon^2\right)$
Skew t	$\lambda Z+Z^{1/2}\varepsilon$	$Ig\left(\nu/2,\nu/2\right)$	$N\left(0,\sigma_\varepsilon^2\right)$
Gen hyperbolic	$\gamma Z+Z^{1/2}\varepsilon$	$GIG\left(\lambda,\chi,\psi\right)$	$N\left(0,\sigma_\varepsilon^2\right)$
Clayton	$(1+\varepsilon/Z)^{-\alpha}$	$\Gamma\left(\alpha,1\right)$	$Exp\left(1\right)$
Gumbel	$-\left(\log Z/\varepsilon\right)^{\alpha}$	$Stable\left(1/\alpha,1,1,0\right)$	$Exp\left(1\right)$

where Ig represents the inverse gamma distribution, GIG is the generalized inverse Gaussian distribution, and Γ is the gamma distribution. The skew t and Generalized hyperbolic copulas listed here are from McNeil, *et al.* (2005, Chapter 5), the representation of a Clayton copula in this form is from Cook and Johnson (1981) and the representation of the Gumbel copula is from Marshall and Olkin (1988).

The above copulas all have closed-form densities via judicious combinations of the "link" function h and the distributions F_z and F_ε. By removing this requirement and employing simulation-based estimation methods to overcome the lack of closed-form likelihood, one can obtain a much wider variety of models for the dependence structure. In this paper we will focus on linear, additive factor copulas, and generate flexible models by flexibly specifying the distribution of the common factor(s).

3 A Monte Carlo study of SMM estimation of factor copulas

As noted above, the class of factor copula models does not generally have a closed-form likelihood, motivating the study of alternative methods for estimation. A general simulation-based method for the estimation of copula models is presented in Oh and Patton (2011), which is ideally suited for the estimation of factor copulas. This estimation method is briefly described in Sections 3.1 and 3.2 below. In Section 3.3 we present an extensive Monte Carlo study of the finite-sample properties of this estimator in applications involving up to 100 variables.

3.1 Description of the model for the conditional joint distribution

We consider the same class of data generating processes (DGPs) as Chen and Fan (2006), Rémillard (2010) and Oh and Patton (2011). This class allows each variable to have time-varying conditional mean and conditional variance, each governed by parametric models, with some unknown marginal distribution. The marginal distributions are estimated nonparametrically via the empirical distribution function. The conditional copula of the data is assumed to belong to a parametric family and is assumed constant, making the model for the joint distribution semiparametric. The combination of time-varying conditional means and variance and a constant conditional copula makes this model similar in spirit to the "CCC" model of Bollerslev (1990). The DGP we consider is:

$$\begin{aligned} \mathbf{Y}_t &= \boldsymbol{\mu}_t(\boldsymbol{\phi}_0) + \boldsymbol{\sigma}_t(\boldsymbol{\phi}_0)\,\boldsymbol{\eta}_t \qquad (16) \\ \text{where } \mathbf{Y}_t &\equiv [Y_{1t}, \ldots, Y_{Nt}]' \\ \boldsymbol{\mu}_t(\boldsymbol{\phi}) &\equiv [\mu_{1t}(\boldsymbol{\phi}), \ldots, \mu_{Nt}(\boldsymbol{\phi})]' \\ \boldsymbol{\sigma}_t(\boldsymbol{\phi}) &\equiv diag\{\sigma_{1t}(\boldsymbol{\phi}), \ldots, \sigma_{Nt}(\boldsymbol{\phi})\} \\ \boldsymbol{\eta}_t &\equiv [\eta_{1t}, \ldots, \eta_{Nt}]' \sim iid \;\; \mathbf{F}_{\boldsymbol{\eta}} = \mathbf{C}(F_1, \ldots, F_N; \boldsymbol{\theta}_0) \end{aligned}$$

where $\boldsymbol{\mu}_t$ and $\boldsymbol{\sigma}_t$ are $\mathcal{F}_{t-1}$-measurable and independent of $\boldsymbol{\eta}_t$. $\mathcal{F}_{t-1}$ is the sigma-field containing information generated by $\{\mathbf{Y}_{t-1}, \mathbf{Y}_{t-2}, \ldots\}$. The $r \times 1$ vector of parameters governing the dynamics of the variables, $\boldsymbol{\phi}_0$, is assumed to be $\sqrt{T}$-consistently estimable. If $\boldsymbol{\phi}_0$ is known, or if $\boldsymbol{\mu}_t$ and $\boldsymbol{\sigma}_t$ are known constant, then the model becomes one for *iid* data. The copula is parameterized by a $p \times 1$ vector of parameters, $\boldsymbol{\theta}_0 \in \boldsymbol{\Theta}$, which is estimated using the SMM approach below.

3.2 Simulation-based estimation of copula models

The simulation-based estimation method of Oh and Patton (2011) is closely related to SMM estimation, though is not strictly SMM, as the "moments" that are used in estimation are functions of rank statistics. Given the similarity, we will nevertheless refer to the method as SMM estimation. Our task is to estimate the $p \times 1$ vector of copula parameters, $\boldsymbol{\theta}_0 \in \boldsymbol{\Theta}$, based on the standardized residual $\left\{\hat{\boldsymbol{\eta}}_t \equiv \boldsymbol{\sigma}_t^{-1}\left(\hat{\boldsymbol{\phi}}\right)\left[\mathbf{Y}_t - \boldsymbol{\mu}_t\left(\hat{\boldsymbol{\phi}}\right)\right]\right\}_{t=1}^{T}$ and simulations from the copula model (for example, the factor copula model in equation 2). The SMM copula estimator of Oh and Patton (2011) is based on simulation from some parametric joint distribution, $\mathbf{F}_x(\boldsymbol{\theta})$, with implied copula $\mathbf{C}(\boldsymbol{\theta})$.

Let $\tilde{\mathbf{m}}_S(\boldsymbol{\theta})$ be a $(m \times 1)$ vector of dependence measures computed using S simulations from

$\mathbf{F}_x(\boldsymbol{\theta})$, $\{\mathbf{X}_s\}_{s=1}^{S}$, and let $\hat{\mathbf{m}}_T$ be the corresponding vector of dependence measures computed using the standardized residuals $\{\hat{\boldsymbol{\eta}}_t\}_{t=1}^{T}$. (We discuss the empirical choice of which dependence measures to match in Appendix B.) The SMM estimator then defined as:

$$\begin{aligned}\hat{\boldsymbol{\theta}}_{T,S} &\equiv \underset{\boldsymbol{\theta}\in\boldsymbol{\Theta}}{\arg\min}\ Q_{T,S}(\boldsymbol{\theta}) \qquad (17)\\ \text{where}\quad Q_{T,S}(\boldsymbol{\theta}) &\equiv \mathbf{g}'_{T,S}(\boldsymbol{\theta})\,\hat{W}_T\mathbf{g}_{T,S}(\boldsymbol{\theta})\\ \mathbf{g}_{T,S}(\boldsymbol{\theta}) &\equiv \hat{\mathbf{m}}_T - \tilde{\mathbf{m}}_S(\boldsymbol{\theta})\end{aligned}$$

and $\hat{W}_T$ is some positive definite weight matrix, which may depend on the data. Under regularity conditions, Oh and Patton (2011) show that if $S/T \to \infty$ as $T \to \infty$, the SMM estimator is consistent and asymptotically normal:[8]

$$\begin{aligned}\sqrt{T}\left(\hat{\boldsymbol{\theta}}_{T,S} - \boldsymbol{\theta}_0\right) &\xrightarrow{d} N(0,\Omega_0) \text{ as } T,S\to\infty \qquad (18)\\ \text{where}\quad \Omega_0 &= \left(G_0'W_0G_0\right)^{-1} G_0'W_0\Sigma_0W_0G_0\left(G_0'W_0G_0\right)^{-1}\end{aligned}$$

$\Sigma_0 \equiv avar[\hat{\mathbf{m}}_T]$, $G_0 \equiv \nabla_\theta \mathbf{g}_0(\boldsymbol{\theta}_0)$, and $\mathbf{g}_0(\boldsymbol{\theta}) = \text{p-lim}_{T,S\to\infty}\ \mathbf{g}_{T,S}(\boldsymbol{\theta})$. Oh and Patton (2011) also present the distribution of a test of the over-identifying restrictions (the "J" test).

The asymptotic variance of the estimator has the same form as in standard GMM applications, however the components Σ_0 and G_0 require different estimation methods than in standard applications. Oh and Patton (2011) show that a simple *iid* bootstrap can be used to consistently estimate Σ_0, and that a standard numerical derivative of $\mathbf{g}_{T,S}(\boldsymbol{\theta})$ at $\hat{\boldsymbol{\theta}}_{T,S}$, denoted $\hat{G}$, will consistently estimate G_0 under the condition that the step size of the numerical derivative goes to zero slower than $T^{-1/2}$. In our simulation study we thoroughly examine the sensitivity of the estimated covariance matrix to the choice of step size.

3.3 Finite-sample properties of SMM estimation of factor copulas

In this section we present a study of the finite sample properties of the simulated method of moments (SMM) estimator of the parameters of various factor copulas. In the case where a likelihood for the copula model is available in closed form we contrast the properties of the SMM estimator with those of the maximum likelihood estimator.

[8] Oh and Patton (2011) also consider the case that $S/T \to 0$ as $S,T\to\infty$, in which case the convergence rate is $\sqrt{S}$ rather than $\sqrt{T}$. In our empirical application we have $S \gg T$, and so we do not present that case here.

3.3.1 Simulation design

We initially consider three different factor copulas, all of them of the form:

$$\begin{aligned}
X_i &= Z + \varepsilon_i, \quad i = 1, 2, ..., N \\
Z &\sim Skew\ t\left(\sigma_z^2, \nu, \lambda\right) \\
\varepsilon_i &\sim iid\ t(\nu), \text{ and } \varepsilon_i \perp\!\!\!\perp Z\ \forall\ i \\
[X_1, ..., X_N]' &\sim \mathbf{F}_x = \mathbf{C}(G_x, ..., G_x)
\end{aligned} \tag{19}$$

and we use the skewed t distribution of Hansen (1994) for the common factor. In all cases we set $\sigma_z^2 = 1$, implying that the common factor (Z) accounts for one-half of the variance of each X_i, implying rank correlation of around 0.5. In the first model we set $\nu \to \infty$ and $\lambda = 0$, which implies that the resulting factor copula is simply the Gaussian copula, with equicorrelation parameter $\rho = 0.5$. In this case we can estimate the model by SMM and also by GMM and MLE, and we use this case to study the loss of efficiency in moving from MLE to GMM to SMM. In the second model we set $\nu = 4$ and $\lambda = 0$, yielding a symmetric factor copula that generates tail dependence. In the third case we set $\nu = 4$ and $\lambda = -0.5$ yielding a factor copula that generates tail dependence as well as "asymmetric dependence", in that the lower tails of the copula are more dependent than the upper tails. We estimate the inverse degrees of freedom parameter, ν_z^{-1}, so that its parameter space is $[0, 0.5)$ rather than $(2, \infty]$.

We also consider an extension of the above equidependence model which allow each X_i to have a different coefficient on Z, as in equation (3). For identification of this model we set $\sigma_z^2 = 1$. For $N = 3$ we set $[\beta_1, \beta_2, \beta_3] = [0.5, 1, 1.5]$. For $N = 10$ we set $[\beta_1, \beta_2, ..., \beta_{10}] = [0.25, 0.50, ..., 2.5]$, which corresponds to pair-wise rank correlations ranging from approximately 0.1 to 0.8. Motivated by our empirical application below, for the $N = 100$ case we consider a "block equidependence" model, where we assume that the 100 variables can be grouped *ex ante* into 10 groups, and that all variables within each group have the same β_i. We use the same set of values for β_i as in the $N = 10$ case.

We consider two different scenarios for the marginal distributions of the variables of interest. In the first case we assume that the data are *iid* with standard Normal marginal distributions, meaning that the only parameters that need to be estimated are those of the factor copula. This simplified case is contrasted with a second scenario where the marginal distributions of the variables are assumed to follow an AR(1)-GARCH(1,1) process:

$$\begin{aligned} Y_{it} &= \phi_0 + \phi_1 Y_{i,t-1} + \sigma_{it}\eta_{it}, \quad t = 1, 2, ..., T \\ \sigma^2_{it} &= \omega + \gamma\sigma^2_{i,t-1} + \alpha\sigma^2_{i,t-1}\eta^2_{i,t-1} \\ \boldsymbol{\eta}_t &\equiv [\eta_{1t}, ..., \eta_{Nt}] \sim iid \ \ \mathbf{F}_\eta = \mathbf{C}(\Phi, \Phi, ..., \Phi) \end{aligned} \tag{20}$$

where Φ is the standard Normal distribution function and $\mathbf{C}$ is the factor copula implied by equation (19). We set the parameters of the marginal distributions as $[\phi_0, \phi_1, \omega, \gamma, \alpha] = [0.01, 0.05, 0.05, 0.85, 0.10]$, which broadly matches the values of these parameters when estimated using daily equity return data. In this scenario the parameters of the marginal distribution are estimated via QML in a separate first stage, following which the estimated standardized residuals, $\hat{\eta}_{it}$, are obtained and used in a second stage to estimate the factor copula parameters. In all cases we consider a time series of length $T = 1000$, corresponding to approximately 4 years of daily return data, and we use $S = 25 \times T$ simulations in the computation of the dependence measures to be matched in the SMM optimization. We repeat each scenario 100 times. In all results below we use the identity weight matrix for estimation; corresponding results based on the efficient weight matrix are available in the web appendix to this paper.[9] In Appendix B we describe the dependence measures we use for the estimation of these models.

3.3.2 Simulation results

Table 1 reveals that for all three dimensions ($N = 3$, 10 and 100) and for all three copula models the estimated parameters are centered on the true values, with the average estimated bias being small relative to the standard deviation, and with the median of the simulated distribution centered on the true values. The measures of estimator accuracy (the standard deviation and the 90-10 percentile difference) reveal that adding more parameters to the model, *ceteris paribus*, leads to greater estimation error, as expected; the σ^2_z parameter, for example, is more accurately estimated when it is the only unknown parameter compared with when it is one of three unknown parameters. Looking across the dimension size, we see that the copula model parameters are almost always more precisely estimated as the dimension grows. This is intuitive, given the equidependence nature of all

[9] The results based on the efficient weight matrix are generally comparable to those based on the identity weight matrix, however the coverage rates are worse than those based on the identity weight matrix.

three models: increasing the dimension of the model does not increase the number of parameters to be estimated but it does increase the amount of information available on the unknown parameters.

Comparing the SMM estimator with the ML estimator, which is only feasible for the Normal copula (as the other two factor copulas do not have a copula likelihood in closed form) we see that the SMM estimator performs quite well. As predicted by theory, the ML estimator is always more efficient than the SMM estimator, however the loss in efficiency is moderate, ranging from around 25% for $N = 3$ to around 10% for $N = 100$. This provides some confidence that our move to SMM, prompted by the lack of a closed-form likelihood, does not come at a cost of a large loss in efficiency. Comparing the SMM estimator to the GMM estimator provides us with a measure of the loss in accuracy from having to estimate the population moment function via simulation. We find that this loss is at most 3% and in some cases ($N = 100$) is slightly negative. Thus little is lost from using SMM rather than GMM when we set $S = 25 \times T$.

Table 2 shows results for the block equidependence model for the $N = 100$ case with AR-GARCH marginal distributions,[10] which can be compared to the results in the lower panel of Table 1. This table shows that the parameters of these models are well estimated using the proposed dependence measures described in Appendix B. The accuracy of the "shape" parameters, ν^{-1} and λ, is slightly lower in the more general model, consistent with the estimation error from having to estimate ten factor loadings (β_i) being greater than from having to estimate just a single other parameter $\left(\sigma_z^2\right)$, however this loss is not great.

[INSERT TABLES 1 AND 2 ABOUT HERE]

In Tables 3 and 4 we present the finite-sample coverage probabilities of 95% confidence intervals based on the estimated asymptotic covariance matrix described in Section 3.2. As discussed above, a critical input to the asymptotic covariance matrix estimator is the step size used in computing the numerical derivative matrix $\hat{G}$. This step size, ε_T, must go to zero, but at a slower rate than $T^{-1/2}$. Ignoring constants, our simulation sample size of $T = 1000$ suggests setting $\varepsilon_T > 0.03$, which is much larger than standard step sizes used in computing numerical derivatives.[11] We consider a range of values from 0.0001 to 0.1. Table 4 shows that when the step size is set to 0.01, 0.03 or

[10] The results for *iid* data, and the results for this model for $N = 3$ and 10, are available in the web appendix.

[11] For example, the default in many *Matlab* functions is a step size of $\varepsilon^{1/3} \approx 6 \times 10^{-6} \approx 1/(165,000)$, where $\varepsilon = 2.22 \times 10^{-16}$ is machine epsilon. This choice is optimal in certain applications, see Judd (1998) for example.

0.1 the finite-sample coverage rates are close to their nominal levels. However if the step size is chosen too small (0.003 or smaller) then the coverage rates are much lower than nominal levels. For example, setting $\varepsilon_T = 0.0001$ (which is still 16 times larger than the default setting in Matlab) we find coverage rates as low as 38% for a nominal 95% confidence interval. Thus this table shows that the asymptotic theory provides a reliable means for obtaining confidence intervals, so long as care is taken not to set the step size too small.

[INSERT TABLES 3 AND 4 ABOUT HERE]

Finally in Table 5 we present the results of a study of the rejection rates for the J test of over-identifying restrictions. Given that we consider $W = I$ in this table, the test statistic has a non-standard distribution (see Proposition 4 of Oh and Patton, 2011), and we use $10,000$ simulations to obtain critical values. In this case, the limiting distribution also depends on $\hat{G}$, and we present the rejection rates for various choices of step size ε_T. Table 5 reveals that the rejection rates are close to their nominal levels, for both the equidependence models and the "different loading" models (which is a block equidependence model for the $N = 100$ case). The J test rejection rates are less sensitive to the choice of step size than the coverage probabilities of confidence intervals, however the best results are again generally obtained when ε_T is 0.01 or greater.

[INSERT TABLE 5 ABOUT HERE]

4 High-dimension copula models for S&P 100 returns

In this section we apply our proposed factor copulas to a study of the dependence between a large collection of U.S. equity returns. We study all 100 stocks that were constituents of the S&P 100 index as at December 2010. The sample period is April 2008 to December 2010, a total of $T = 696$ trade days. The starting point for our sample period was determined by the date of the latest addition to the S&P 100 index (Philip Morris Inc.), which has had no additions or deletions since April 2008. The stocks in our study are listed in Table 6, along with their 3-digit SIC codes, which we will use in part of our analysis below.

[INSERT TABLE 6 ABOUT HERE]

Table 7 presents some summary statistics of the data used in this analysis. The top panel presents sample moments of the daily returns for each stock. The means and standard deviations are around values observed in other studies. The skewness and kurtosis coefficients reveal a substantial degree of heterogeneity in the shape of the distribution of these asset returns, motivating our use of a nonparametric estimate (the EDF) of this in our analysis.

In the second panel of Table 7 we present information on the parameters of the AR(1)–GJR-GARCH models, augmented with lagged market return information, that are used to filter each of the individual return series[12]:

$$r_{it} = \phi_{0i} + \phi_{1i} r_{i,t-1} + \phi_{mi} r_{m,t-1} + \varepsilon_{it} \tag{21}$$

$$\begin{aligned} \sigma_{it}^2 &= \omega_i + \beta_i \sigma_{i,t-1}^2 + \alpha_i \varepsilon_{i,t-1}^2 + \gamma_i \varepsilon_{i,t-1}^2 \mathbf{1}\{\varepsilon_{i,t-1} \leq 0\} \\ &\quad + \alpha_{mi} \varepsilon_{m,t-1}^2 + \gamma_{mi} \varepsilon_{m,t-1}^2 \mathbf{1}\{\varepsilon_{m,t-1} \leq 0\} \end{aligned} \tag{22}$$

As in our simulation study, we estimate the parameters of the mean and variance models using QML, and we estimate the distribution of the standardized residuals using the empirical distribution function (EDF). The use of the EDF allows us to nonparametrically capture skewness and excess kurtosis in the residuals, if present, and allows these characteristics to differ across the 100 variables.

Our estimates of the parameters of these models are consistent with those reported in numerous other studies, with a small negative AR(1) coefficient found for most though not all stocks, and with the lagged market return entering significantly in 37 out of the 100 stocks. The estimated GJR-GARCH parameters are strongly indicative of persistence in volatility, and the asymmetry parameter, γ, in this model is positive for all but three of the 100 stocks in our sample, supporting the wide-spread finding of a "leverage effect" in the conditional volatility of equity returns. The lagged market residual is also found to be important for volatility in many cases, with the null that $\alpha_{mi} = \gamma_{mi} = 0$ being rejected at the 5% level for 32 stocks.

In the lower panel of Table 7 we present summary statistics for four measures of dependence between pairs of standardized residuals: linear correlation, rank correlation, average upper and lower 1% tail dependence (equal to $(\tau_{0.99} + \tau_{0.01})/2$), and the difference in upper and lower 10% tail dependence (equal to $\tau_{0.90} - \tau_{0.10}$). The two correlation statistics measure the sign and strength

[12] We considered GARCH (Bollerslev, 1986), EGARCH (Nelson, 1991), and GJR-GARCH (Glosten, *et al.*, 1993) models for the conditional variance of these returns, and for almost all stocks the GJR-GARCH model was preferred according to the BIC.

of dependence, the third and fourth statistics measure the strength and symmetry of dependence in the tails. The two correlation measures are similar, and are 0.42 and 0.44 on average. Across all 4950 pairs of assets the rank correlation varies from 0.37 to 0.50 from the 25^{th} and 75^{th} percentiles of the cross-sectional distribution, indicating the presence of mild heterogeneity in the correlation coefficients. The 1% tail dependence measure is 0.06 on average, and varies from 0.00 to 0.07 across the inter-quartile range. The difference in the 10% tail dependence measures is negative on average, and indeed is negative for over 75% of the pairs of stocks, strongly indicating asymmetric dependence between these stocks.

[INSERT TABLE 7 ABOUT HERE]

4.1 Results from equidependence copula specifications

We now present our first empirical results on the dependence structure of these 100 stock returns: the estimated parameters of eight different models for the copula. We consider four existing copulas: the Clayton copula, the Normal copula, the Student's t copula, and the skew t copula, with equicorrelation imposed on the latter three models for comparability, and four factor copulas, described by the distributions assumed for the common factor and the idiosyncractic shock: t-Normal, Skew t-Normal, t-t, Skew t-t. All models are estimated using the SMM-type method described in Section 3.2. The value of the SMM objective function at the estimated parameters, Q_{SMM}, is presented for each model, along with the p-value from the J-test of the over-identifying restrictions. Standard errors are based on 1000 bootstraps to estimate $\Sigma_{T,S}$, and with a step size $\varepsilon_T = 0.1$ to compute $\hat{G}$.

Table 8 reveals that the variance of the common factor, σ_z^2, is estimated by all models to be around 0.9, implying an average correlation coefficient of around 0.47. The estimated inverse degrees of freedom parameter in these models is around 1/25, and the standard errors on ν^{-1} reveal that this parameter is significant[13] at the 10% level for the three models that allow for asymmetric dependence, but not significant for the three models that impose symmetric dependence. The asymmetry parameter, λ, is significantly negative in all models in which it is estimated, with t-

[13] Note that the case of zero tail dependence corresponds to $\nu_z^{-1} = 0$, which is on the boundary of the parameter space, implying that a standard t test is strictly not applicable. In such cases the squared t statistic no longer has an asymptotic χ_1^2 distribution under the null, rather it is distributed as an equal-weighted mixture of a χ_1^2 and χ_0^2, see Gourieroux and Monfort (1996, Ch 21). The 90% and 95% critical values for this distribution are 1.64 and 2.71, which correspond to t-statistics of 1.28 and 1.65.

statistics ranging from -2.1 to -4.4. This implies that the dependence structure between these stock returns is significantly asymmetric, with large crashes being more likely than large booms. Other papers have considered equicorrelation models for the dependence between large collections of stocks, see Engle and Kelly (2012) for example, but empirically showing the importance of allowing the implied common factor to be fat tailed and asymmetric is novel.

[INSERT TABLE 8 ABOUT HERE]

Figure 4 presents the quantile dependence function from the estimated Normal copula and the estimated skew $t-t$ factor copula, along with the quantile dependence averaged across all pairs of stocks, and pointwise 90% bootstrap confidence intervals for these estimates based on the theory in Rémillard (2010). (The figure zooms in on the left and right 20% tails, removing the middle 60% of the distribution as the estimates and models are all very similar there.) This figure reveals that the Normal copula overestimates the dependence in the upper tail, and underestimates it in the lower tail. This is consistent with the fact that the empirical quantile dependence is asymmetric, while the Normal copula imposes symmetry. The skew $t-t$ factor copula provides a reasonable fit in both tails, though it somewhat overestimates the dependence in the extreme left tail.

Figure 5 exploits the high-dimensional nature of our analysis, and plots the expected proportion of "crashes" in the remaining $(100-j)$ stocks, conditional on observing a crash in j stocks. We show this for a "crash" defined as a once-in-a-month (1/22, around 4.6%) event and as a once-in-a-quarter (1/66, around 1.5%) event. For once-in-a-month crashes, the observed proportions track the Skew t-t factor copula well for j up to around 25 crashes, and again for j of around 70. For j in between 30 and 65 the Normal copula appears to fit quite well. For once-in-a-quarter crashes, displayed in the lower panel of Figure 5, the empirical plot tracks that for the Normal copula well for j up to around 30, but for $j=35$ the empirical plot jumps and follows the skew $t-t$ factor copula. Thus it appears that the Normal copula may be adequate for modeling moderate tail events, but a copula with greater tail dependence (such as the skew $t-t$ factor copula) is needed for more extreme tail events.

[INSERT FIGURES 4 AND 5 ABOUT HERE]

The last two columns of Table 8 report the value of the objective function (Q_{SMM}) and the p-value from a test of the over-identifying restrictions. The Q_{SMM} values reveal that the three models that allow for asymmetry (skew t copula, and the two skew t factor copulas) out-perform all the other models, and reinforce the above conclusion that allowing for a skewed common factor is important for this collection of assets. The p-values, however, are near zero for all models, indicating that none of them pass this specification test. One likely source of these rejections is the assumption of equidependence, which was shown in the summary statistics in Table 7 to be questionable for this large set of stock returns. We relax this in the next section.

4.2 Results from *block* equidependence copula specifications

In response to the rejection of the copula models based on equidependence, we now consider a generalization to allow for heterogeneous dependence. We propose a multi-factor model that allows for a common, market-wide, factor, and a set of factors related only to specific industries. We use the first digit of Standard Industrial Classification (SIC) to form seven groups of stocks, see Table 6. The model we consider is the copula generated by the following structure:

$$\begin{aligned} X_i &= \beta_i Z_0 + \gamma_i Z_{S(i)} + \varepsilon_i, \;\; i = 1, 2, ..., 100 \\ Z_0 &\sim Skew\ t(\nu, \lambda) \\ Z_S &\sim iid\ t(\nu), \;\; S = 1, 2, ..., 7; \;\; Z_S \perp\!\!\!\perp Z_0 \;\forall\; S \\ \varepsilon_i &\sim iid\ t(\nu), \;\; i = 1, 2, ..., 100; \;\; \varepsilon_i \perp\!\!\!\perp Z_j \;\forall\; i, j \end{aligned} \tag{23}$$

where $S(i)$ is the SIC group for stock i. There are eight latent common factors in total in this model, but any given variable is only affected by two factors, simplifying its structure and reducing the number of free parameters. Note here we impose that the industry factors and the idiosyncratic shocks are symmetric, and only allow asymmetry in the market-wide factor, Z_0. It is feasible to consider allowing the industry factors to have differing levels of asymmetry, but we rule this out in the interests of parsimony. We impose that all stocks in the same SIC group have the same factor loadings, but allow stocks in different groups to have different factor loadings. This generates a "block equidependence" model which greatly increases the flexibility of the model, but without generating too many additional parameters to estimate. In total, this copula model has a total of 16 parameters, providing more flexibility than the 3-parameter equidependence model considered in the previous section, but still more parsimonious (and tractable) than a completely unstructured approach to this 100-dimensional problem.[14]

The results of this model are presented in Table 9. The Clayton copula is not presented here as it imposes equidependence by construction, and so is not comparable to the other models. The estimated inverse degrees of freedom parameter, ν^{-1}, is around 1/14, which is larger and more significant than for the equidependence model, indicating stronger evidence of tail dependence. The asymmetry parameters are also larger (in absolute value) and more significantly negative in this more flexible model than in the equidependence model. It appears that when we add variables that control for intra-industry dependence, (i.e., industry-specific factors) we find the market-wide common factor is more fat tailed and left skewed than when we impose a single factor structure.

[INSERT TABLE 9 ABOUT HERE]

Focusing on our preferred *skew* $t-t$ factor copula model, the coefficients on the market factor, β_i, range from 0.88 (for SIC group 2, Manufacturing: Food, apparel, etc.) to 1.25 (SIC group 1, Mining and construction), indicating the varying degrees of inter-industry dependence. The coefficients on the industry factors, γ_i, measure the degree of additional intra-industry dependence, beyond that coming from the market-wide factor. These range from 0.17 to 1.09 for SIC groups 3 and 1 respectively. Even for the smaller estimates, these are significantly different from zero, indicating the presence of industry factors beyond a common market factor. The intra- and inter-industry rank correlations and tail dependence coefficients implied by this model[15] are presented in Table 10, and reveal the degree of heterogeneity and asymmetry that this copula captures: rank correlations range from 0.39 (for pairs of stocks in SIC groups 1 and 5) to 0.72 (for stocks within SIC group 1). The upper and lower tail dependence coefficients further reinforce the importance of asymmetry in the dependence structure, with lower tail dependence measures being substantially larger than upper tail measures: lower tail dependence averages 0.82 and ranges from 0.70 to 0.99, while upper tail dependence averages 0.07 and ranges from 0.02 to 0.74.

[INSERT TABLE 10 ABOUT HERE]

[14] We also considered a one-factor model that allowed for different factor loadings, generalizing the equidependence model of the previous section but simpler than this multi-factor copula model. That model provided a significantly better fit than the equidependence model, but was also rejected using the J test of over-identifying restrictions, and so is not presented here to conserve space.

[15] Rank correlations from this model are not available in closed form, and we use 50,000 simulations to estimate these. Upper and lower tail dependence coefficients are based on Propositions 2 and 3.

With this more flexible model we can test restrictions on the factor coefficients, to see whether the additional flexibility is required to fit the data. The p-values from these tests are in the bottom rows of Table 9. Firstly, we can test whether all of the industry factor coefficients are zero, which reduces this model to a one-factor model with flexible weights. The p-values from these tests are zero to four decimal places for all models, providing strong evidence in favor of including industry factors. We can also test whether the market factor is needed given the inclusion of industry factors by testing whether all betas are equal to zero, and predictably this restriction is strongly rejected by the data. We further can test whether the coefficients on the market and industry factors are common across all industries, reducing this model to an equidependence model, and this too is strongly rejected. Finally, we use the J test of over-identifying restrictions to check the specification of these models. Using this test, we see that the models that impose symmetry are strongly rejected. The skew t copula has a p-value of 0.04, indicating a marginal rejection, and the *skew* $t - t$ factor copula performs best, passing this test at the 5% level, with a p-value of 0.07.

Thus it appears that a multi-factor model with heterogeneous weights on the factors, that allows for positive tail dependence and stronger dependence in crashes than booms, is needed to fit the dependence structure of these 100 stock returns.

4.3 Measuring systemic risk: Marginal Expected Shortfall

The recent financial crisis has highlighted the need for the management and measurement of systemic risk, see Acharya *et al.* (2010) for discussion. Brownlees and Engle (2011) propose a measure of systemic risk they call "marginal expected shortfall", or MES. It is defined as the expected return on stock i given that the market return is below some (low) threshold:

$$MES_{it} = -E_{t-1}\left[r_{it} | r_{mt} < C\right] \tag{24}$$

An appealing feature of this measure of systemic risk is that it can be computed with only a bivariate model for the conditional distribution of (r_{it}, r_{mt}), and Brownlees and Engle (2011) propose a semiparametric model based on a bivariate DCC-GARCH model to estimate it. A corresponding drawback of this measure is that by using a market index to identify periods of crisis, it may overlook periods with crashes in individual firms. With a model for the entire set of constituent stocks, such as the high dimension copula models considered in this paper, combined with standard AR-GARCH type models for the marginal distributions, we can estimate the MES measure proposed in Brownlees and Engle (2011), as well as alternative measures that use crashes in individual stocks as flags for periods of turmoil. For example, one might consider the expected return on stock i conditional on k stocks in the market having returns below some threshold, a "kES":

$$kES_{it} = -E_{t-1}\left[r_{it}\left|\left(\sum\nolimits_{j=1}^{N}\mathbf{1}\left\{r_{jt} < C\right\}\right) > k\right.\right] \tag{25}$$

Brownlees and Engle (2011) propose a simple method for ranking estimates of MES:

$$\begin{aligned} MSE_i &= \frac{1}{T}\sum_{t=1}^{T}\left(r_{it} - MES_{it}\right)^2\mathbf{1}\left\{r_{mt} < C\right\} \\ RelMSE_i &= \frac{1}{T}\sum_{t=1}^{T}\left(\frac{r_{it} - MES_{it}}{MES_{it}}\right)^2\mathbf{1}\left\{r_{mt} < C\right\} \end{aligned} \tag{26}$$

Corresponding metrics immediately follow for estimates of "kES".

In Table 11 we present the MSE and RelMSE for estimates of MES and kES, for threshold choices of -2% and -4%. We implement the model proposed by Brownlees and Engle (2011), as well as their implementations of a model based on the CAPM, and one based purely on rolling historical information. Along with these, we present results for four copulas: the Normal, Student's t, skew t, and skew $t-t$ factor copula, all with the block equidependence structure from Section 4.2 above. In the upper panel of Table 11 we see that the Brownlees-Engle model performs the best for both thresholds under the MSE performance metric, with the skew $t-t$ factor copula as the second-best performing model. Under the Relative MSE metric, the factor copula is best performing model, for both thresholds, followed by the skew t copula. Like Brownlees and Engle (2011), we find that the worst-performing methods under both metrics are the Historical and CAPM methods.

The lower panel of Table 11 presents the performance of various methods for estimating kES, with k set to 30.[16] This measure requires an estimate of the conditional distribution for the entire set of 100 stocks, and thus the CAPM and Brownlees-Engle methods cannot be applied. We evaluate the remaining five methods, and find that the skew $t-t$ factor copula performs the best for both thresholds, under both metrics. Thus our proposed factor copula model for high dimensional dependence allows us to gain some insights into the structure of the dependence between this large collection of assets, and also provides improved estimates of measures of systemic risk.

[INSERT TABLE 11 ABOUT HERE]

[16] We choose this value of k so that the number of identified "crisis" days is broadly comparable to the number of such days for MES. Results for alternative values of k are similar.

5 Conclusion

This paper presents new models for the dependence structure, or copula, of economic variables based on a simple factor structure for the copula. The proposed models are particularly attractive for high dimensional applications, involving fifty or more variables, as they allow the researcher to increase or decrease the flexibility of the model according to the amount of data available and the dimension of the problem, and, importantly, to do so in a manner that is easily interpreted. The class of factor copulas presented in this paper does not generally have a closed-form likelihood. We use extreme value theory to obtain analytical results on the tail dependence implied by factor copulas, and we consider SMM-type methods for the estimation of factor copulas. Via an extensive Monte Carlo study, we show that SMM estimation has good finite-sample properties in time series applications involving up to 100 variables.

We employ our proposed factor copulas to study daily returns on all 100 constituents of the S&P 100 index over the period 2008-2010, and find significant evidence of a skewed, fat-tailed common factor, which generates asymmetric dependence and tail dependence. In an extension to a multi-factor copula, we find evidence of the importance of industry factors, leading to heterogeneous dependence. We also consider an application to the estimation of systemic risk, and we show that the proposed factor copula model provides superior estimates of two measures of systemic risk.

Appendix A: Proofs

Proof of Proposition 1. Consider a simple case first: $\beta_1 = \beta_2 = \beta > 0$. This implies that $X_i \sim G$, for $i = 1, 2$, and so we can use the same threshold for both X_1 and X_2. Then the upper tail dependence coefficient is:

$$\tau^U = \lim_{s\to\infty} \frac{\Pr[X_1 > s, X_2 > s]}{\Pr[X_1 > s]}$$

From standard extreme value theory, see Hyung and de Vries (2007) for example, we have the probability of an exceedence by the sum as the sum of the probabilities of an exceedence by each component of the sum, as the exceedence threshold diverges:

$$\begin{aligned}\Pr[X_i > s] &= \Pr[\beta Z + \varepsilon_i > s] \\ &= \Pr[\beta Z > s] + \Pr[\varepsilon_i > s] + o\left(s^{-\alpha}\right) \quad \text{as } s \to \infty \\ &\approx A_z^U (s/\beta)^{-\alpha} + A_\varepsilon^U s^{-\alpha} \\ &= s^{-\alpha}\left(A_z^U \beta^\alpha + A_\varepsilon^U\right)\end{aligned}$$

Further, we have the probability of *two* sums of variables both exceeding some diverging threshold being driven completely be the common component of the sums:

$$\begin{aligned}\Pr[X_1 > s, X_2 > s] &= \Pr[\beta Z + \varepsilon_1 > s, \beta Z + \varepsilon_2 > s] \\ &= \Pr[\beta Z > s, \beta Z > s] + o\left(s^{-\alpha}\right) \quad \text{as } s \to \infty \\ &\approx s^{-\alpha} A_z^U \beta^\alpha\end{aligned}$$

So we have

$$\tau^U = \lim_{s\to\infty} \frac{s^{-\alpha} A_z^U \beta^\alpha}{s^{-\alpha}\left(A_z^U \beta^\alpha + A_\varepsilon^U\right)} = \frac{A_z^U \beta^\alpha}{A_z^U \beta^\alpha + A_\varepsilon^U}$$

(a) Now we consider the case that $\beta_1 \neq \beta_2$, and *wlog* assume $\beta_2 > \beta_1 > 0$. This complicates the problem as the thresholds, s_1 and s_2, must be set such that $G_1(s_1) = G_2(s_2) = q \to 1$, and when $\beta_1 \neq \beta_2$ we have $G_1 \neq G_2$ and so $s_1 \neq s_2$. We can find the link between the thresholds as follows:

$$\Pr[X_i > s] = \Pr[\beta_i Z + \varepsilon_i > s] \approx s^{-\alpha}\left(A_z^U \beta_i^\alpha + A_\varepsilon^U\right) \quad \text{for } s \to \infty$$

so find s_1, s_2 such that $s_1^{-\alpha}\left(A_z^U \beta_1^\alpha + A_\varepsilon^U\right) = s_2^{-\alpha}\left(A_z^U \beta_2^\alpha + A_\varepsilon^U\right)$, which implies:

$$s_2 = s_1 \left(\frac{A_z^U \beta_2^\alpha + A_\varepsilon^U}{A_z^U \beta_1^\alpha + A_\varepsilon^U}\right)^{1/\alpha}$$

Note that s_1 and s_2 diverge at the same rate. Below we will need to know which of s_1/β_1 and s_2/β_2 is larger. Note that $\beta_2 > \beta_1$, which implies the following:

$$\begin{aligned}&\Rightarrow \quad \beta_2^\alpha > \beta_1^\alpha \quad \text{since } x^\alpha \text{ is increasing in } x \text{ for } x, \alpha > 0 \\ &\Rightarrow \quad A_\varepsilon^U \beta_2^\alpha + A_z^U \beta_1^\alpha \beta_2^\alpha > A_\varepsilon^U \beta_1^\alpha + A_z^U \beta_1^\alpha \beta_2^\alpha \\ &\Rightarrow \quad \left(\frac{\beta_2}{\beta_1}\right)^\alpha > \frac{A_\varepsilon^U + A_z^U \beta_2^\alpha}{A_\varepsilon^U + A_z^U \beta_1^\alpha} \\ &\Rightarrow \quad \frac{\beta_2}{\beta_1} > \left(\frac{A_\varepsilon^U + A_z^U \beta_2^\alpha}{A_\varepsilon^U + A_z^U \beta_1^\alpha}\right)^{1/\alpha} = \frac{s_2}{s_1} \\ &\Rightarrow \quad \frac{s_1}{\beta_1} > \frac{s_2}{\beta_2}\end{aligned}$$

Then the denominator of the tail dependence coefficient is $\Pr[X_i > s_i] \approx s_i^{-\alpha}\left(A_z^U \beta_i^\alpha + A_\varepsilon^U\right)$, and the numerator becomes:

$$\begin{aligned}
\Pr[X_1 > s_1, X_2 > s_2] &= \Pr[\beta_1 Z + \varepsilon_1 > s_1, \beta_2 Z + \varepsilon_2 > s_2] \\
&\approx \Pr[\beta_1 Z > s_1, \beta_2 Z > s_2] \quad \text{as } s_1, s_2 \to \infty \\
&= \Pr[Z > \max\{s_1/\beta_1, s_2/\beta_2\}] \\
&= \Pr[Z > s_1/\beta_1] = s_1^{-\alpha} A_z^U \beta_1^\alpha
\end{aligned}$$

Finally, using either $\Pr[X_1 > s_1]$ or $\Pr[X_2 > s_2]$ in the denominator we obtain:

$$\tau^U = \frac{s_1^{-\alpha} A_z^U \beta_1^\alpha}{s_1^{-\alpha}\left(A_z^U \beta_1^\alpha + A_\varepsilon^U\right)} = \frac{\beta_1^\alpha A_z^U}{\beta_1^\alpha A_z^U + A_\varepsilon^U}, \text{ as claimed.}$$

(b) Say $\beta_2 < \beta_1 < 0$. Then:

$$\begin{aligned}
\Pr[X_i > s] &= \Pr[\beta_i Z + \varepsilon_i > s] \\
&\approx \Pr[\beta_i Z > s] + \Pr[\varepsilon_i > s] \quad \text{for } s \to \infty \\
&= \Pr[|\beta_i|(-Z) > s] + \Pr[\varepsilon_i > s] \\
&= s^{-\alpha}\left(A_z^L |\beta_i|^\alpha + A_\varepsilon^U\right)
\end{aligned}$$

Next we find the thresholds s_1, s_2 such that $\Pr[X_1 > s_1] = \Pr[X_2 > s_2]$:

$$\begin{aligned}
s_1^{-\alpha}\left(A_z^L |\beta_1|^\alpha + A_\varepsilon^U\right) &= s_2^{-\alpha}\left(A_z^L |\beta_2|^\alpha + A_\varepsilon^U\right) \\
\text{so } s_2 &= s_1 \left(\frac{A_z^L |\beta_2|^\alpha + A_\varepsilon^U}{A_z^L |\beta_1|^\alpha + A_\varepsilon^U}\right)^{1/\alpha}
\end{aligned}$$

Using the same steps as for part (a), we find that $s_2 > s_1$ but $s_1/|\beta_1| > s_2/|\beta_2|$. Thus the numerator becomes:

$$\begin{aligned}
\Pr[X_1 > s_1, X_2 > s_2] &= \Pr[\beta_1 Z + \varepsilon_1 > s_1, \beta_2 Z + \varepsilon_2 > s_2] \\
&\approx \Pr[\beta_1 Z > s_1, \beta_2 Z > s_2] \quad \text{for } s_1, s_2 \to \infty \\
&= \Pr[|\beta_1|(-Z) > s_1, |\beta_2|(-Z) > s_2] \\
&= \Pr[(-Z) > \max\{s_1/|\beta_1|, s_2/|\beta_2|\}] \\
&= \Pr[(-Z) > s_1/|\beta_1|] = A_z^L s_1^{-\alpha} |\beta_1|^\alpha \\
\text{so } \tau^U &= \frac{|\beta_1|^\alpha A_z^L}{|\beta_1|^\alpha A_z^L + A_\varepsilon^U}
\end{aligned}$$

(c) If β_1 or β_2 equal zero, then the numerator of the upper tail dependence coefficient limits to zero faster than the denominator. Say $\beta_2 > \beta_1 = 0:$

$$\begin{aligned}
\Pr[X_1 > s_1] &= s_1^{-\alpha}\left(A_z^U \beta_1^\alpha + A_\varepsilon^U\right) = s_1^{-\alpha} A_\varepsilon^U = \mathcal{O}\left(s^{-\alpha}\right) \\
\text{and} \quad \Pr[X_2 > s_2] &= s_2^{-\alpha}\left(A_z^U \beta_2^\alpha + A_\varepsilon^U\right) = \mathcal{O}\left(s^{-\alpha}\right) \\
\text{but} \quad \Pr[X_1 > s_1, X_2 > s_2] &= \Pr[\varepsilon_1 > s_1, \beta_2 Z + \varepsilon_2 > s_2] \\
&= \Pr[\varepsilon_1 > s_1] \Pr[\beta_2 Z + \varepsilon_2 > s_2] \\
&= A_\varepsilon^U s_1^{-\alpha}\left(A_z^U \beta_2^\alpha + A_\varepsilon^U\right) s_2^{-\alpha} \quad \text{as } s \to \infty \\
&= \mathcal{O}\left(s^{-2\alpha}\right) \\
\text{so} \quad \frac{\Pr[X_1 > s_1, X_2 > s_2]}{\Pr[X_1 > s_1]} &= \mathcal{O}\left(s^{-\alpha}\right) \to 0 \quad \text{as } s \to \infty.
\end{aligned}$$

(d) Say $\beta_1 < 0 < \beta_2$. Then the denominator will be order $\mathcal{O}(s^{-\alpha})$, but the numerator will be of a lower order:

$$\begin{aligned}
\Pr[X_1 > s_1, X_2 > s_2] &= \Pr[\beta_1 Z + \varepsilon_1 > s_1, \beta_2 Z + \varepsilon_2 > s_2] \\
&= \Pr[\beta_1 Z > s_1, \beta_2 Z > s_2] + o\left(s^{-\alpha}\right) \quad \text{as } s \to \infty \\
&= o\left(s^{-\alpha}\right)
\end{aligned}$$

since $\Pr[\beta_1 Z > s_1, \beta_2 Z > s_2] = 0$ as $s_1, s_2 > 0$ $(\to \infty)$ and $sgn(\beta_1 Z) = -sgn(\beta_2 Z)$. Thus $\tau^U = o(s^{-\alpha}) / \mathcal{O}(s^{-\alpha}) = o(1) \to 0$ as $s \to \infty$. All of the results for parts (a) through (d) apply for lower tail dependence, *mutatis mutandis.* ■

Proof of Proposition 2. It is more convenient to work with the density than the distribution function for skew t random variables. Note that if F_z has a regularly varying tails with tail index $\alpha > 0$, then

$$\begin{aligned}
F_z(s) &\equiv \Pr[Z \leq s] = 1 - \Pr[Z > s] = 1 - A_z^U s^{-\alpha} \quad \text{as} \quad s \to \infty \\
f_z(s) &\equiv \frac{\partial F_z(s)}{\partial s} = -\frac{\partial}{\partial s} \Pr[Z > s] = \alpha A_z^U s^{-\alpha-1} \quad \text{as} \quad s \to \infty \\
\text{so} \quad A_z^U &= \lim_{s \to \infty} \frac{f_z(s)}{\alpha s^{-\alpha-1}}
\end{aligned}$$

This representation of the extreme tails of a density function is common in EVT, see Embrechts, *et al.* (1997) and Daníelsson, *et al.* (2012) for example. For $\nu \in (2, \infty)$ and $\lambda \in (-1, 1)$, the skew

t distribution of Hansen (1994) has density:

$$f_z(s;\nu,\lambda) = \begin{cases} bc\left(1+\frac{1}{\nu-2}\left(\frac{bz+a}{1-\lambda}\right)^2\right)^{-(\nu+1)/2}, & z<-a/b \\ bc\left(1+\frac{1}{\nu-2}\left(\frac{bz+a}{1+\lambda}\right)^2\right)^{-(\nu+1)/2}, & z\geq -a/b \end{cases}$$

$$\text{where } a = 4\lambda c\left(\frac{\nu-2}{\nu-1}\right),\ b=\sqrt{1+3\lambda^2-a^2},\ c=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi(\nu-2)}}$$

and its tail index is equal to the degrees of freedom parameter, so $\alpha=\nu$. Using computational algebra software such as Mathematica, it is possible to show that

$$A_z^U = \lim_{s\to\infty}\frac{f_z(s)}{\nu s^{-\nu-1}} = \frac{bc}{\nu}\left(\frac{b^2}{(\nu-2)(1+\lambda)^2}\right)^{-(\nu+1)/2}$$

For the left tail we have

$$\begin{aligned} f_z(s) &\equiv \frac{\partial F_z(s)}{\partial s} = \frac{\partial}{\partial s}A_z^L(-s)^{-\alpha} \quad \text{as } s\to-\infty \\ &= \alpha A_z^L(-s)^{-\alpha-1} \\ \text{and so } A_z^L &= \lim_{s\to-\infty}\frac{f_z(s)}{\nu(-s)^{-\nu-1}} \end{aligned}$$

And this can be shown to equal

$$A_z^L = \lim_{s\to-\infty}\frac{f_z(s)}{\alpha(-s)^{-\alpha-1}} = \frac{bc}{\nu}\left(\frac{b^2}{(\nu-2)(1-\lambda)^2}\right)^{-(\nu+1)/2}$$

When $\lambda=0$ we recover the non-skewed, standardized Student's t distribution. In that case we have $a=0$, $b=1$ (and c unchanged), and so $A_\varepsilon^U = A_\varepsilon^L = \frac{c}{\nu}\left(\frac{1}{\nu-2}\right)^{-(\nu+1)/2}$. ■

Proof of Proposition 3. First consider the denominator of the upper tail dependence coefficient:

$$\begin{aligned} \Pr[X_i>s_i] &= \Pr\left[\sum_{k=1}^K \beta_{ik}Z_k+\varepsilon_i>s_i\right] \\ &\approx \Pr[\varepsilon_i>s_i]+\sum_{k=1}^K \Pr[\beta_{ik}Z_k>s_i] \quad \text{for } s_i\to\infty \\ &= s_i^{-\alpha}\left(A_\varepsilon^U+\sum_{k=1}^K A_k^U\beta_{ik}^\alpha\right) \end{aligned}$$

We need to choose $s_i, s_j\to\infty$ such that $\Pr[X_i>s_i]=\Pr[X_j>s_j]$, which implies

$$s_j = s_i\left(\frac{A_\varepsilon^U+\sum_{k=1}^K A_k^U\beta_{jk}^\alpha}{A_\varepsilon^U+\sum_{k=1}^K A_k^U\beta_{ik}^\alpha}\right)^{1/\alpha} \equiv s_i\gamma_{U,ij}$$

As in Proposition 1, note that s_i and s_j diverge at the same rate.

When $\beta_{ik}\beta_{jk} = 0$, the factor Z_k does not contribute to the numerator of the tail dependence coefficient, as it appears in at most one of X_i and X_j. Thus we need only keep track of factors such that $\beta_{ik}\beta_{jk} > 0$. In this case, we again need to determine the larger of s_i/β_{ik} and s_j/β_{jk} for each $k = 1, 2, ..., K$. Unlike the one-factor model, a general ranking cannot be obtained. To keep notation compact we introduce δ_{ijk}. Note

$$\begin{aligned} \max\left\{\frac{s_i}{\beta_{ik}}, \frac{s_j}{\beta_{jk}}\right\} &= \max\left\{\frac{s_i}{\beta_{ik}}, \frac{s_i}{\beta_{jk}}\gamma_{U,ij}\right\} = \frac{s_i}{\beta_{ij}} \max\left\{1, \frac{\beta_{ik}}{\beta_{jk}}\gamma_{U,ij}\right\} \equiv \frac{s_i}{\beta_{ik}\delta_{ijk}} \\ \text{where } \delta_{ijk}^{-1} &\equiv \max\left\{1, \frac{\beta_{ik}}{\beta_{jk}}\gamma_{U,ij}\right\} \end{aligned}$$

To cover the case that $\beta_{ik}\beta_{jk} = 0$, we generalize the definition of δ_{ijk} so that it is well defined in that case. The use of any finite number here will work (as it will be multiplied by zero in this case) and we set it to one:

$$\delta_{ijk}^{-1} \equiv \begin{cases} \max\left\{1, \gamma_{U,ij}\beta_{ik}/\beta_{jk}\right\}, & \text{if } \beta_{ik}\beta_{jk} > 0 \\ 1, & \text{if } \beta_{ik}\beta_{jk} = 0 \end{cases}$$

Now we can consider the numerator

$$\begin{aligned} \Pr\left[X_i > s_i, X_j > s_j\right] &= \Pr\left[\sum\nolimits_{k=1}^{K} \beta_{ik}Z_k + \varepsilon_i > s_i, \sum\nolimits_{k=1}^{K} \beta_{jk}Z_k + \varepsilon_j > s_j\right] \\ &\approx \sum\nolimits_{k=1}^{K} \Pr\left[\beta_{ik}Z_k > s_i, \beta_{jk}Z_k > s_j\right] \quad \text{for } s_i, s_k \to \infty \end{aligned}$$

$$\begin{aligned} &= \sum\nolimits_{k=1}^{K} \mathbf{1}\left\{\beta_{ik}\beta_{jk} > 0\right\} \Pr\left[\beta_{ik}Z_k > s_i, \beta_{jk}Z_k > s_j\right] \\ &= \sum\nolimits_{k=1}^{K} \mathbf{1}\left\{\beta_{ik}\beta_{jk} > 0\right\} \Pr\left[Z_k > \max\left\{\frac{s_i}{\beta_{ik}}, \frac{s_j}{\beta_{jk}}\right\}\right] \\ &\equiv \sum\nolimits_{k=1}^{K} \mathbf{1}\left\{\beta_{ik}\beta_{jk} > 0\right\} \Pr\left[Z_k > \frac{s_i}{\beta_{ik}\delta_{ijk}}\right] \\ &= s_i^{-\alpha} \sum\nolimits_{k=1}^{K} \mathbf{1}\left\{\beta_{ik}\beta_{jk} > 0\right\} A_k^U \beta_{ik}^{\alpha} \delta_{ijk}^{\alpha} \end{aligned}$$

And so we obtain

$$\tau_{ij}^U = \lim_{s\to\infty} \frac{\Pr\left[X_i > s_i, X_j > s_j\right]}{\Pr\left[X_i > s_i\right]} = \frac{\sum_{k=1}^{K} \mathbf{1}\left\{\beta_{ik}\beta_{jk} > 0\right\} A_k^U \beta_{ik}^{\alpha} \delta_{ijk}^{\alpha}}{A_\varepsilon^U + \sum_{k=1}^{K} A_k^U \beta_{ik}^{\alpha}}$$

The results for lower tail dependence can be obtained using similar derivations to those above. ■

Appendix B: Choice of dependence measures for estimation

To implement the SMM estimator of these copula models we must first choose which dependence measures to use in the SMM estimation. We draw on "pure" measures of dependence, in the sense that they are solely affected by changes in the copula, and not by changes in the marginal distributions. For examples of such measures, see Joe (1997, Chapter 2) or Nelsen (2006, Chapter 5). Our preliminary studies of estimation accuracy and identification lead us to use pair-wise rank correlation, and quantile dependence with $q = [0.05, 0.10, 0.90, 0.95]$, giving us five dependence measures for each pair of variables.

Let δ_{ij} denote one of the dependence measures (i.e., rank correlation or quantile dependence at different levels of q) between variables i and j, and define the "pair-wise dependence matrix":

$$D = \begin{bmatrix} 1 & \delta_{12} & \cdots & \delta_{1N} \\ \delta_{12} & 1 & \cdots & \delta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{1N} & \delta_{2N} & \cdots & 1 \end{bmatrix} \tag{27}$$

Where applicable, we exploit the (block) equidependence feature of the models in defining the "moments" to match. For the initial set of simulation results and for the first model in the empirical section, the model implies equidependence, and we use as "moments" the average of these five dependence measures across all pairs, reducing the number of moments to match from $5N(N-1)/2$ to just 5:

$$\bar{\delta} \equiv \frac{2}{N(N-1)} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \hat{\delta}_{ij} \tag{28}$$

For a model with different loadings on the common factor (as in equation 3) equidependence does not hold. Yet the common factor aspect of the model implies that there are $\mathcal{O}(N)$, not $\mathcal{O}(N^2)$, parameters driving the pair-wise dependence matrices. In light of this, we use the $N \times 1$ vector $\left[\bar{\delta}_1, ..., \bar{\delta}_N\right]'$, where

$$\bar{\delta}_i \equiv \frac{1}{N} \sum_{j=1}^{N} \hat{\delta}_{ij}$$

and so $\bar{\delta}_i$ is the average of all pair-wise dependence measures that involve variable i. This yields a total of $5N$ moments for estimation.

For the block-equidependence version of this model (used for the $N = 100$ case in the simulation, and in the second set of models for the empirical section), we exploit the fact that (i) all variables in the same group exhibit equidependence, and (ii) any pair of variables (i, j) in groups (r, s) has the same dependence as any other pair (i', j') in the same two groups (r, s). This allows us to average all intra- and inter-group dependence measures. Consider the following general design, where we have N variables, M groups, and k_m variables per group, where $\Sigma_{m=1}^{M} k_m = N$. Then decompose the $(N \times N)$ matrix D into sub-matrices according to the groups:

$$\underset{(N\times N)}{D} = \begin{bmatrix} D_{11} & D'_{12} & \cdots & D'_{1M} \\ D_{12} & D_{22} & \cdots & D'_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ D_{1M} & D_{2M} & \cdots & D_{MM} \end{bmatrix}, \text{ where } D_{ij} \text{ is } (k_i \times k_j) \tag{29}$$

Then create a matrix of average values from each of these matrices, taking into account the fact that the diagonal blocks are symmetric:

$$\underset{(M\times M)}{D^*} = \begin{bmatrix} \delta^*_{11} & \delta^*_{12} & \cdots & \delta^*_{1m} \\ \delta^*_{12} & \delta^*_{22} & \cdots & \delta^*_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^*_{1m} & \delta^*_{2m} & \cdots & \delta^*_{mm} \end{bmatrix} \tag{30}$$

$$\begin{aligned} \text{where } \delta^*_{ss} &\equiv \frac{2}{k_s(k_s-1)} \sum\sum \hat{\delta}_{ij}, \text{ avg of } \textit{all upper triangle} \text{ values in } D_{ss} \\ \delta^*_{rs} &= \frac{1}{k_r k_s} \sum\sum \hat{\delta}_{ij}, \text{ avg of } \textit{all} \text{ elements in matrix } D_{rs}, \quad r \neq s \end{aligned}$$

Finally, similar to the previous model, create the vector of average measures $\left[\bar{\delta}^*_1, ..., \bar{\delta}^*_M\right]$, where

$$\bar{\delta}^*_i \equiv \frac{1}{M} \sum_{j=1}^{M} \delta^*_{ij} \tag{31}$$

This gives as a total of M moments for each dependence measure, so $5M$ in total.

References

[1] Aas, K., C. Czado, A. Frigessi and H. Bakken, 2009, Pair-copula constructions of multiple dependence, *Insurance: Mathematics and Economics*, 44, 182-198.

[2] Acar, E.F., C. Genest and J. Nešlehová, 2012, Beyond simplified pair-copula constructions, *Journal of Multivariate Analysis*, 110, 74-90.

[3] Andersen, L. and J. Sidenius, 2004, Extensions to the Gaussian copula: random recovery and random factor loadings, *Journal of Credit Risk*, 1, 29-70.

[4] Barndorff-Nielsen, O.E., 1978, Hyperbolic distributions and distributions on hyperbolae, *Scandinavian Journal of Statistics*, 5, 151-157.

[5] Barndorff-Nielsen, O.E., 1997, Normal inverse Gaussian distributions and stochastic volatility modelling, *Scandinavian Journal of Statistics*, 24, 1-13.

[6] Barndorff-Nielsen, O.E. and N. Shephard, 2009, *Financial Volatility in Continuous Time*, Cambridge University Press, Cambridge, forthcoming.

[7] Bollerslev, T., 1986, Generalized autoregressive conditional heteroskedasticity, *Journal of Econometrics*, 31, 307-327.

[8] Bollerslev, T., 1990, Modelling the coherence in short-run nominal exchange rates: a multivariate generalized ARCH approach, *Review of Economics and Statistics*, 72, 498-505.

[9] Chen, X. and Y. Fan, 2006, Estimation and model selection of semiparametric copula-based multivariate dynamic models under copula misspecification, *Journal of Econometrics*, 135, 125-154.

[10] Cherubini, U., E. Luciano and W. Vecchiato, 2004, *Copula Methods in Finance*, John Wiley & Sons, England.

[11] Christoffersen, P., V. Errunza, K. Jacobs and H. Langlois, 2011, Is the potential for international diversification disappearing?, *Review of Financial Studies*, forthcoming.

[12] Cook, R.D. and M.E. Johnson, 1981, A family of distributions for modelling non-elliptically symmetric multivariate data, *Journal of the Royal Statistical Society*, 43, 210-218.

[13] Coval, J., J. Jurek and E. Stafford, 2009, The economics of structured finance, *Journal of Economic Perspectives*, 23, 3-25.

[14] Daníelsson, J., B.N. Jorgensen, G. Samorodnitsky, M. Sarma and C.G. de Vries, 2012, Fat tails, VaR and subadditivity, *Journal of Econometrics*, forthcoming.

[15] Daul, S., E. De Giorgi, F. Lindskog and A. McNeil, 2003, The grouped t-copula with an application to credit risk, *RISK*, 16, 73-76.

[16] Duffie, D., A. Eckner, G. Horel and L. Saita, 2009, Frailty correlated default, *Journal of Finance*, 64(5), 2089-2123.

[17] Demarta, S. and A.J. McNeil, 2005, The t copula and related copulas, *International Statistical Review*, 73, 111-129.

[18] Embrechts, P., C. Klüppelberg and T. Mikosch, 1997, *Modelling Extremal Events*, Springer-Verlag, Berlin.

[19] Engle, R.F. and B. Kelly, 2012, Dynamic Equicorrelation, *Journal of Business and Economic Statistics*, 30(2), 212-228.

[20] Engle, R.F., N. Shephard and K. Sheppard, 2008, Fitting vast dimensional time-varying covariance models, working paper, Oxford-Man Institute, University of Oxford.

[21] Fan, J., Y. Fan and J. Lv, 2008, High dimensional covariance matrix estimation using a factor model, *Journal of Econometrics*, 147, 186-197.

[22] Fan, J., Y. Li and Y. Ke, 2012, Vast volatility matrix estimation using high frequency data for portfolio selection, *Journal of American Statistical Association*, 107, 412-428.

[23] Geluk, J.L., L. de Haan and C.G. de Vries, 2007, Weak and Strong Financial Fragility, Tinbergen Institute Discussion Paper TI 2007-023/2.

[24] Glosten, L.R., R. Jagannathan, D.E. Runkle, 1993, On the relation between the expected value and the volatility of the nominal excess return on stocks, *Journal of Finance*, 48(5), 1779-1801.

[25] Gouriéroux, C. and A. Monfort, 1996, *Statistics and Econometric Models, Volume 2*, translated from the French by Q. Vuong, Cambridge University Press, Cambridge.

[26] Hansen, B.E., 1994, Autoregressive conditional density estimation, *International Economic Review*, 35(3), 705-730.

[27] Hautsch, N., L.M. Kyj and R.C.A. Oomen, 2012, A blocking and regularization approach to high dimensional realized covariance estimation, *Journal of Applied Econometrics*, 27, 625-645.

[28] Heinen, A., and A. Valdesogo, 2009, Asymmetric CAPM dependence for large dimensions: the canonical vine autoregressive model, working paper, Univerisidad Carlos III.

[29] Hull, J. and A. White, 2004, Valuation of a CDO and an n^{th} to default CDS without Monte Carlo simulation, *Journal of Derivatives*, 12, 8-23.

[30] Hyung, N. and C.G. de Vries, 2007, Portfolio selection with heavy tails, *Journal of Empirical Finance*, 14, 383-400.

[31] Joe, H., 1997, *Multivariate Models and Dependence Concepts*, Monographs in Statistics and Probability 73, Chapman and Hall, London.

[32] Judd, K.L, 1998, *Numerical Methods in Economics*, MIT Press, Cambridge, USA.

[33] Li, D.X., 2000, On default correlation: A copula function approach, *Journal of Fixed Income*, 9, 43-54.

[34] Marshall, A.W. and I. Olkin, 1988, Families of multivariate distributions, *Journal of the American Statistical Association*, 83, 834-841.

[35] McNeil, A.J., R. Frey and P. Embrechts, 2005, *Quantitative Risk Management*, Princeton University Press, New Jersey.

[36] Min, A. and C. Czado, 2010, Bayesian inference for multivariate copulas using pair-copula constructions, *Journal of Financial Econometrics*, 8, 511-546.

[37] Nelsen, R.B., 2006, *An Introduction to Copulas*, Second Edition, Springer, U.S.A.

[38] Nelson, D.B., 1991, Conditional heteroskedasticity in asset returns: A new approach, *Econometrica*, 59, 347-370.

[39] Oh, D.-H. and A.J. Patton, 2011, Simulated method of moments estimation for copula-based multivariate models, Working paper, Duke University.

[40] Palm, F.C. and J.-P. Urbain, 2011, Factor structures for panel and multivariate time series data, *Journal of Econometrics*, 163(1), 1-3.

[41] Patton, A.J., 2006, Modelling asymmetric exchange rate dependence, *International Economic Review*, 47(2), 527-556.

[42] Patton, A.J., 2012, Copula methods for forecasting multivariate time series, in G. Elliott and A. Timmermann (eds.), *Handbook of Economic Forecasting*, Volume 2, Elsevier, Oxford, forthcoming.

[43] Rémillard, B., 2010, Goodness-of-fit tests for copulas of multivariate time series, working paper.

[44] Smith, M., Q. Gan and R. Kohn, 2012, Modeling dependence using skew t copulas: Bayesian inference and applications, *Journal of Applied Econometrics*, 27, 500-522.

[45] van der Voort, M., 2005, Factor copulas: totally external defaults, working paper, Erasmus University Rotterdam.

[46] White, H., 1994, *Estimation, Inference and Specification Analysis*, Econometric Society Monographs No. 22, Cambridge University Press, Cambridge, U.K.

[47] Zimmer, D.M., 2012, The role of copulas in the housing crisis, *Review of Economics and Statistics*, 94, 607-620.

Table 1: Simulation results for factor copula models

	Normal MLE	Normal GMM	Normal SMM	Factor $t-t$		Factor *skew* $t-t$		
	σ_z^2	σ_z^2	σ_z^2	σ_z^2	ν^{-1}	σ_z^2	ν^{-1}	λ
True value	1.00	1.00	1.00	1.00	0.25	1.00	0.25	-0.50
				$N=3$				
Bias	0.0141	-0.0143	-0.0164	-0.0016	-0.0185	0.0126	-0.0199	-0.0517
Std	0.0803	0.1014	0.1033	0.1094	0.0960	0.1205	0.1057	0.1477
Median	1.0095	0.9880	0.9949	0.9956	0.2302	1.0050	0.2380	-0.5213
90%	1.1180	1.1103	1.1062	1.1448	0.3699	1.1772	0.3636	-0.3973
10%	0.9172	0.8552	0.8434	0.8721	0.0982	0.8662	0.0670	-0.7538
90-10 Diff	0.2008	0.2551	0.2628	0.2727	0.2716	0.3110	0.2966	0.3565
				$N=10$				
Bias	0.0113	-0.0099	-0.0119	-0.0025	-0.0137	-0.0039	-0.0161	-0.0119
Std	0.0559	0.0651	0.0666	0.0724	0.0611	0.0851	0.0790	0.0713
Median	1.0125	0.9874	0.9898	0.9926	0.2360	0.9897	0.2376	-0.5084
90%	1.0789	1.0644	1.0706	1.0967	0.3102	1.1095	0.3420	-0.4318
10%	0.9406	0.9027	0.8946	0.9062	0.1704	0.8996	0.1331	-0.5964
90-10 Diff	0.1383	0.1617	0.1761	0.1905	0.1398	0.2100	0.2089	0.1645
				$N=100$				
Bias	0.0167	-0.0068	-0.0080	-0.0011	-0.0138	0.0015	-0.0134	-0.0099
Std	0.0500	0.0554	0.0546	0.0659	0.0549	0.0841	0.0736	0.0493
Median	1.0164	0.9912	0.9956	1.0011	0.2346	0.9943	0.2402	-0.5101
90%	1.0805	1.0625	1.0696	1.0886	0.3127	1.1060	0.3344	-0.4465
10%	0.9534	0.9235	0.9279	0.9112	0.1685	0.8970	0.1482	-0.5734
90-10 Diff	0.1270	0.1390	0.1418	0.1773	0.1442	0.2089	0.1861	0.1270

Notes: This table presents the results from 100 simulations of three different factor copulas, the Normal copula, the $t-t$ factor copula and the *skew* $t-t$ factor copula. The Normal copula is estimated by ML, GMM, and SMM, and the other two copulas are estimated by SMM. The marginal distributions of the data are assumed to follow AR(1)-GARCH(1,1) processes, as described in Section 3. Problems of dimension $N=3$, 10 and 100 are considered, the sample size is $T=1000$ and the number of simulations used is $S=25\times T$. The first row of each panel presents the average difference between the estimated parameter and its true value. The second row presents the standard deviation in the estimated parameters. The third, fourth and fifth rows present the 50^{th}, 90^{th} and 10^{th} percentiles of the distribution of estimated parameters, and the final row presents the difference between the 90^{th} and 10^{th} percentiles.

Table 2: Simulation results for different loadings factor copula model with N=100

	ν^{-1}	λ_z	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
True value	0.25	-0.5	0.25	0.5	0.75	1	1.25	1.5	1.75	2	2.25	2.5
						Normal						
Bias	-	-	-0.0010	-0.0038	-0.0040	-0.0072	-0.0071	-0.0140	-0.0178	-0.0119	-0.0194	-0.0208
Std	-	-	0.0128	0.0182	0.0248	0.0322	0.0377	0.0475	0.0651	0.0784	0.1022	0.1291
Median	-	-	0.2489	0.4970	0.7440	0.9942	1.2421	1.4868	1.7279	1.9918	2.2256	2.4832
90%	-	-	0.2645	0.5204	0.7787	1.0291	1.2970	1.5470	1.8226	2.0874	2.3609	2.6458
10%	-	-	0.2304	0.4701	0.7158	0.9502	1.1982	1.4197	1.6526	1.8825	2.0921	2.3090
90-10 diff	-	-	0.0341	0.0503	0.0629	0.0788	0.0987	0.1273	0.1700	0.2049	0.2689	0.3368
						Factor $t-t$						
Bias	-0.0120	-	0.0000	0.0009	0.0018	-0.0045	0.0011	-0.0073	-0.0080	-0.0122	-0.0061	-0.0065
Std	0.0574	-	0.0149	0.0236	0.0300	0.0343	0.0443	0.0580	0.0694	0.0867	0.1058	0.1332
Median	0.2384	-	0.2503	0.5056	0.7528	0.9985	1.2550	1.4881	1.7409	1.9820	2.2234	2.4737
90%	0.3056	-	0.2678	0.5255	0.7896	1.0348	1.3052	1.5697	1.8270	2.1012	2.4089	2.6597
10%	0.1683	-	0.2348	0.4689	0.7187	0.9462	1.1965	1.4282	1.6517	1.8744	2.1303	2.3196
90-10 diff	0.1373	-	0.0330	0.0566	0.0709	0.0886	0.1086	0.1416	0.1754	0.2268	0.2786	0.3401
						Factor *skew* $t-t$						
Bias	-0.0119	-0.0019	0.0008	0.0001	0.0028	-0.0029	-0.0036	-0.0096	-0.0114	-0.0232	-0.0178	-0.0194
Std	0.0633	0.0451	0.0134	0.0246	0.0320	0.0443	0.0588	0.0806	0.0902	0.1111	0.1373	0.1635
Median	0.2434	-0.5051	0.2477	0.5001	0.7520	0.9986	1.2468	1.4826	1.7417	1.9803	2.2107	2.4786
90%	0.3265	-0.4392	0.2680	0.5309	0.7961	1.0613	1.3028	1.5856	1.8378	2.1094	2.4430	2.7034
10%	0.1550	-0.5527	0.2358	0.4660	0.7155	0.9505	1.1756	1.4042	1.6230	1.8395	2.0494	2.2739
90-10 diff	0.1714	0.1134	0.0321	0.0648	0.0807	0.1107	0.1272	0.1814	0.2148	0.2699	0.3936	0.4294

Notes: This table presents the results from 100 simulations of three different factor copulas: the Normal copula, the $t-t$ factor copula and the *skew* $t-t$ factor copula. We divide the $N=100$ variables into ten groups and assume that all variables in the same group have the same loading on the common factor. The marginal distributions of the data are assumed to follow AR(1)-GARCH(1,1) processes, as described in Section 3. The sample size is $T=1000$ and the number of simulations used is $S=25\times T$. The first row of each panel presents the average difference between the estimated parameter and its true value. The second row presents the standard deviation in the estimated parameters. The third, fourth and fifth rows present the 50^{th}, 90^{th} and 10^{th} percentiles of the distribution of estimated parameters, and the final row presents the difference between the 90^{th} and 10^{th} percentiles.

Table 3: Simulation results on coverage rates

	Normal	Factor $t-t$		Factor *skew* $t-t$		
	σ_z^2	σ_z^2	ν^{-1}	σ_z^2	ν^{-1}	λ
	$N=3$					
ε_T						
0.1	89	93	97	99	100	96
0.03	90	94	98	99	98	96
0.01	88	92	98	99	96	95
0.003	85	95	95	96	89	95
0.001	83	89	89	92	84	93
0.0003	58	69	69	74	74	74
0.0001	38	49	53	57	70	61
	$N=10$					
ε_T						
0.1	87	93	99	97	98	99
0.03	87	95	99	97	98	97
0.01	87	94	96	97	98	95
0.003	87	95	95	98	95	96
0.001	87	95	93	96	90	95
0.0003	86	94	87	91	77	93
0.0001	71	87	81	71	81	85
	$N=100$					
ε_T						
0.1	95	93	95	94	95	94
0.03	95	94	94	94	94	94
0.01	95	93	93	94	94	94
0.003	94	95	93	94	94	94
0.001	94	94	92	94	93	95
0.0003	92	94	92	94	92	93
0.0001	84	94	89	94	88	95

Notes: This table presents the results from 100 simulations of three different factor copulas, the Normal copula, the $t-t$ factor copula and the *skew* $t-t$ factor copula, all estimated by SMM. The marginal distributions of the data are assumed to follow AR(1)-GARCH(1,1) processes, as described in Section 3. Problems of dimension $N=3$, 10 and 100 are considered, the sample size is $T=1000$ and the number of simulations used is $S=25\times T$. The rows of each panel contain the step size, ε_T, used in computing the matrix of numerical derivatives, $\hat{G}_{T,S}$. The numbers in the table present the percentage of simulations for which the 95% confidence interval based on the estimated covariance matrix contained the true parameter.

Table 4: Coverage rate for different loadings factor copula model with N=100 AR-GARCH data

	ν^{-1}	λ	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
						Normal						
ε_T												
0.1	-	-	97	91	92	89	95	93	94	95	95	90
0.03	-	-	97	91	92	90	95	95	94	95	95	90
0.01	-	-	97	91	92	90	95	94	94	96	94	91
0.003	-	-	97	90	93	90	95	94	95	96	95	90
0.001	-	-	97	90	94	93	94	94	94	96	94	92
0.0003	-	-	97	92	93	92	95	94	91	93	92	94
0.0001	-	-	94	94	91	88	90	92	94	91	88	86
						Factor $t-t$						
ε_T												
0.1	95	-	94	93	96	96	98	91	93	92	95	93
0.03	94	-	94	91	96	96	98	92	93	92	97	93
0.01	95	-	94	94	97	96	97	93	93	92	98	93
0.003	94	-	94	94	97	96	97	94	94	95	98	95
0.001	94	-	93	93	97	97	97	92	96	94	100	94
0.0003	90	-	94	95	98	97	99	94	95	95	99	93
0.0001	65	-	95	96	96	98	98	92	96	94	97	91
						Factor *skew* $t-t$						
ε_T												
0.1	93	95	98	95	96	94	94	92	91	91	90	92
0.03	93	95	98	95	95	94	95	92	91	91	89	90
0.01	93	95	97	96	95	94	94	92	92	91	91	91
0.003	93	95	97	96	96	94	95	92	92	92	90	89
0.001	93	94	97	96	95	94	94	91	91	93	89	88
0.0003	84	93	98	95	95	95	95	90	90	88	83	85
0.0001	69	86	98	97	94	91	90	88	87	84	83	80

Notes: This table presents the results from 100 simulations of three different factor copulas: the Normal copula, the $t-t$ factor copula and the *skew* $t-t$ factor copula. We divide the $N = 100$ variables into ten groups and assume that all variables in the same group have the same loading on the common factor. The marginal distributions of the data are assumed to follow AR(1)-GARCH(1,1) processes, as described in Section 3. The sample size is $T = 1000$ and the number of simulations used is $S = 25 \times T$. The rows of each panel contain the step size, ε_T, used in computing the matrix of numerical derivatives, $\hat{G}_{T,S}$. The numbers in the table present the percentage of simulations for which the 95% confidence interval based on the estimated covariance matrix contained the true parameter.

Table 5: Rejection frequencies for the test of overidentifying restrictions

	Equidependence			Different loadings		
	Normal	Factor $t-t$	Factor *skew* $t-t$	Normal	Factor $t-t$	Factor *skew* $t-t$
			$N=3$			
ε_T						
0.1	97	97	99	95	97	97
0.03	97	98	99	95	95	96
0.01	97	97	100	93	95	95
0.003	97	98	100	92	95	96
0.001	98	96	100	93	93	97
0.0003	99	97	100	91	92	97
0.0001	99	97	99	92	94	98
			$N=10$			
ε_T						
0.1	97	97	98	98	95	98
0.03	98	97	97	98	95	99
0.01	96	97	97	97	94	98
0.003	97	96	97	98	92	99
0.001	98	95	97	96	89	100
0.0003	97	94	97	97	93	100
0.0001	97	94	98	98	95	100
			$N=100$			
ε_T						
0.1	97	95	99	95	95	99
0.03	97	95	98	96	94	99
0.01	97	95	98	96	93	99
0.003	97	95	97	95	94	99
0.001	97	94	99	95	91	100
0.0003	97	94	99	95	89	100
0.0001	98	92	98	93	90	100

Notes: This table presents the results from 100 simulations of three different factor copulas, the Normal copula, the $t-t$ factor copula and the *skew* $t-t$ factor copula, all estimated by SMM. The marginal distributions of the data are assumed to follow AR(1)-GARCH(1,1) processes, as described in Section 3. Problems of dimension $N=3$, 10 and 100 are considered, the sample size is $T=1000$ and the number of simulations used is $S=25\times T$. The rows of each panel contain the step size, ε_T, used in computing the matrix of numerical derivatives, $\hat{G}_{T,S}$, needed for the critical value. The confidence level for the test of over-identifying restrictions is 0.95, and the numbers in the table present the percentage of simulations for which the test statistic was greater than its computed critical value.

Table 6: **Stocks used in the empirical analysis**

Ticker	Name	SIC	Ticker	Name	SIC	Ticker	Name	SIC
AA	Alcoa	333	EXC	Exelon	493	NKE	Nike	302
AAPL	Apple	357	F	Ford	371	NOV	National Oilwell	353
ABT	Abbott Lab.	283	FCX	Freeport	104	NSC	Norfolk Sth	671
AEP	American Elec	491	FDX	Fedex	451	NWSA	News Corp	271
ALL	Allstate Corp	633	GD	GeneralDynam	373	NYX	NYSE Euronxt	623
AMGN	Amgen Inc.	283	GE	General Elec	351	ORCL	Oracle	737
AMZN	Amazon.com	737	GILD	GileadScience	283	OXY	OccidentalPetrol	131
AVP	Avon	284	GOOG	Google Inc	737	PEP	Pepsi	208
AXP	American Ex	671	GS	GoldmanSachs	621	PFE	Pfizer	283
BA	Boeing	372	HAL	Halliburton	138	PG	Procter&Gamble	284
BAC	Bank of Am	602	HD	Home Depot	525	QCOM	Qualcomm Inc	366
BAX	Baxter	384	HNZ	Heinz	203	RF	Regions Fin	602
BHI	Baker Hughes	138	HON	Honeywell	372	RTN	Raytheon	381
BK	Bank of NY	602	HPQ	HP	357	S	Sprint	481
BMY	Bristol-Myers	283	IBM	IBM	357	SLB	Schlumberger	138
BRK	Berkshire Hath	633	INTC	Intel	367	SLE	Sara Lee Corp.	203
C	Citi Group	602	JNJ	Johnson&J.	283	SO	Southern Co.	491
CAT	Caterpillar	353	JPM	JP Morgan	672	T	AT&T	481
CL	Colgate	284	KFT	Kraft	209	TGT	Target	533
CMCSA	Comcast	484	KO	Coca Cola	208	TWX	Time Warner	737
COF	Capital One	614	LMT	Lock'dMartn	376	TXN	Texas Inst	367
COP	Conocophillips	291	LOW	Lowe's	521	UNH	UnitedHealth	632
COST	Costco	533	MA	Master card	615	UPS	United Parcel	451
CPB	Campbell	203	MCD	MaDonald	581	USB	US Bancorp	602
CSCO	Cisco	367	MDT	Medtronic	384	UTX	United Tech	372
CVS	CVS	591	MET	Metlife Inc.	671	VZ	Verizon	481
CVX	Chevron	291	MMM	3M	384	WAG	Walgreen	591
DD	DuPont	289	MO	Altria Group	211	WFC	Wells Fargo	602
DELL	Dell	357	PM	Philip Morris	211	WMB	Williams	492
DIS	Walt Disney	799	MON	Monsanto	287	WMT	WalMart	533
DOW	Dow Chem	282	MRK	Merck	283	WY	Weyerhauser	241
DVN	Devon Energy	131	MS	MorganStanley	671	XOM	Exxon	291
EMC	EMC	357	MSFT	Microsoft	737	XRX	Xerox	357
ETR	ENTERGY	491						

	Description	Num		Description	Num
SIC 1	Mining, construct.	6	SIC 5	Trade	8
SIC 2	Manuf: food, furn.	26	SIC 6	Finance, Ins	18
SIC 3	Manuf: elec, mach	25	SIC 7	Services	6
SIC 4	Transprt, comm's	11	ALL		100

Notes: This table presents the ticker symbols, names and 3-digit SIC codes of the 100 stocks used in the empirical analysis of this paper. The lower panel reports the number of stocks in each 1-digit SIC group.

Table 7: Summary statistics

	Cross-sectional distribution					
	Mean	5%	25%	Median	75%	95%
Mean	0.0004	-0.0003	0.0001	0.0003	0.0006	0.0013
Std dev	0.0287	0.0153	0.0203	0.0250	0.0341	0.0532
Skewness	0.3458	-0.4496	-0.0206	0.3382	0.6841	1.2389
Kurtosis	11.3839	5.9073	7.5957	9.1653	11.4489	19.5939
ϕ_0	0.0004	-0.0004	0.0001	0.0004	0.0006	0.0013
ϕ_1	-0.0345	-0.2045	-0.0932	-0.0238	0.0364	0.0923
ϕ_m	-0.0572	-0.2476	-0.1468	-0.0719	0.0063	0.1392
$\omega \times 1000$	0.0126	0.0024	0.0050	0.0084	0.0176	0.0409
β	0.8836	0.7983	0.8639	0.8948	0.9180	0.9436
α	0.0240	0.0000	0.0000	0.0096	0.0354	0.0884
γ	0.0593	0.0000	0.0017	0.0396	0.0928	0.1628
α_m	0.0157	0.0000	0.0000	0.0000	0.0015	0.0646
γ_m	0.1350	0.0000	0.0571	0.0975	0.1577	0.3787
ρ	0.4155	0.2643	0.3424	0.4070	0.4749	0.5993
ρ_s	0.4376	0.2907	0.3690	0.4292	0.4975	0.6143
$(\tau_{0.99} + \tau_{0.01})/2$	0.0572	0.0000	0.0000	0.0718	0.0718	0.1437
$(\tau_{0.90} - \tau_{0.10})$	-0.0922	-0.2011	-0.1293	-0.0862	-0.0431	0.0144

Notes: This table presents some summary statistics of the daily equity returns data used in the empirical analysis. The top panel presents simple unconditional moments of the daily return series. The second panel presents summaries of the estimated AR(1)–GJR-GARCH(1,1) models estimated on these returns. The lower panel presents linear correlation, rank correlation, average 1% upper and lower tail dependence, and the difference between the 10% tail dependence measures, computed using the standardized residuals from the estimated AR–GJR-GARCH model. The columns present the mean and quantiles from the cross-sectional distribution of the measures listed in the rows. The top two panels present summaries across the $N = 100$ marginal distributions, while the lower panel presents a summary across the $N(N-1)/2 = 4950$ distinct pairs of stocks.

Table 8: Estimation results for daily returns on S&P 100 stocks

	σ_z^2		ν^{-1}		λ		Q_{SMM}	p-val
	Est	Std Err	Est	Std Err	Est	Std Err		
Clayton†	0.6017	0.0345	–	–	–	–	0.0449	0.0000
Normal	0.9090	0.0593	–	–	–	–	0.0090	0.0000
Student's t	0.8590	0.0548	0.0272	0.0292	–	–	0.0119	0.0000
Skew t	0.6717	0.0913	0.0532	0.0133	-8.3015	4.0202	0.0010	0.0020
Factor $t-N$	0.8978	0.0555	0.0233	0.0325	–	–	0.0101	0.0000
Factor $skew\ t-N$	0.8954	0.0565	0.0432	0.0339	-0.2452	0.0567	0.0008	0.0002
Factor $t-t$	0.9031	0.0591	0.0142	0.0517	–	–	0.0098	0.0000
Factor $skew\ t-t$	0.8790	0.0589	0.0797	0.0486	-0.2254	0.0515	0.0007	0.0005

Notes: This table presents estimation results for various copula models applied to 100 daily stock returns over the period April 2008 to December 2010. Estimates and asymptotic standard errors for the copula model parameters are presented, as well as the value of the SMM objective function at the estimated parameters and the p-value of the overidentifying restriction test. Note that the parameter λ lies in $(-1, 1)$ for the factor copula models, but in $(-\infty, \infty)$ for the Skew t copula; in all cases the copula is symmetric when $\lambda = 0$. †Note that the parameter of the Clayton copula is not σ_z^2 but we report it in this column for simplicity.

Table 9: Estimation results for daily returns on S&P 100 stocks for block equidependence copula models

	Normal		Student's t		Skew t		Factor $t-t$		Factor *skew* $t-t$	
	Est	Std Err	Est	Std Err	Est	Std Err	Est	Std Err	Est	Std Err
ν^{-1}	-	-	0.0728	0.0269	0.0488	0.0069	0.0663	0.0472	0.0992	0.0455
λ	-	-	-	-	-9.6597	1.0860	-	-	-0.2223	0.0550
β_1	1.3027	0.0809	1.2773	0.0754	1.1031	0.1140	1.2936	0.0800	1.2457	0.0839
β_2	0.8916	0.0376	0.8305	0.0386	0.7343	0.0733	0.8528	0.0362	0.8847	0.0392
β_3	0.9731	0.0363	0.9839	0.0380	0.9125	0.0638	1.0223	0.0389	1.0320	0.0371
β_4	0.9426	0.0386	0.8751	0.0367	0.7939	0.0715	0.9064	0.0375	0.9063	0.0415
β_5	1.0159	0.0555	0.9176	0.0523	0.8171	0.0816	0.9715	0.0527	0.9419	0.0551
β_6	1.1018	0.0441	1.0573	0.0435	0.9535	0.0733	1.0850	0.0441	1.0655	0.0457
β_7	1.0954	0.0574	1.0912	0.0564	1.0546	0.0839	1.1057	0.0535	1.1208	0.0601
γ_1	1.0339	0.0548	0.9636	0.0603	1.0292	0.0592	1.0566	0.0595	1.0892	0.0582
γ_2	0.4318	0.0144	0.3196	0.0388	0.3474	0.0411	0.3325	0.0215	0.2201	0.0457
γ_3	0.4126	0.0195	0.3323	0.0394	0.2422	0.0458	0.2147	0.0727	0.1701	0.0996
γ_4	0.4077	0.0235	0.3726	0.0328	0.3316	0.0289	0.3470	0.0273	0.2740	0.0495
γ_5	0.4465	0.0403	0.5851	0.0300	0.5160	0.0324	0.5214	0.0312	0.5459	0.0448
γ_6	0.6122	0.0282	0.5852	0.0351	0.5581	0.0286	0.5252	0.0274	0.5686	0.0407
γ_7	0.5656	0.0380	0.5684	0.0464	0.2353	0.0889	0.3676	0.0434	0.3934	0.0548
Q_{SMM}	0.1587		0.1567		0.0266		0.1391		0.0189	
J *p-value*	0.0000		0.0000		0.0437		0.0000		0.0722	
$\gamma_i = 0\ \forall i$	0.0000		0.0000		0.0000		0.0000		0.0000	
$\beta_i = 0\ \forall i$	0.0000		0.0000		0.0000		0.0000		0.0000	
$\beta_i = \beta_j, \gamma_i = \gamma_j\ \forall i,j$	0.0000		0.0000		0.0000		0.0000		0.0000	

Notes: This table presents estimation results for various block equidependence copula models applied to filtered daily returns on collections of 100 stocks over the period April 2008 to December 2010. Estimates and asymptotic standard errors for the model parameters are presented. Note that the parameter λ lies in $(-1, 1)$ for the factor copula models, but in $(-\infty, \infty)$ for the Skew t copula; in all cases the copula is symmetric when $\lambda = 0$. The bottom three rows present p-values from tests of constraints on the coefficients on the factors.

Table 10: Rank correlation and tail dependence implied by a multi-factor model

	SIC 1	SIC 2	SIC 3	SIC 4	SIC 5	SIC 6	SIC 7
				Rank correlation			
SIC 1	0.72						
SIC 2	0.41	0.44					
SIC 3	0.44	0.45	0.51				
SIC 4	0.41	0.42	0.45	0.46			
SIC 5	0.39	0.40	0.44	0.41	0.53		
SIC 6	0.42	0.43	0.47	0.43	0.42	0.58	
SIC 7	0.45	0.46	0.50	0.46	0.44	0.47	0.57
				Lower \ Upper tail dependence			
SIC 1	0.99 \ 0.74	0.02	0.07	0.02	0.03	0.09	0.13
SIC 2	0.70	0.70 \ 0.02	0.02	0.02	0.02	0.02	0.02
SIC 3	0.92	0.70	0.92 \ 0.07	0.02	0.03	0.07	0.07
SIC 4	0.75	0.70	0.75	0.75 \ 0.02	0.02	0.02	0.02
SIC 5	0.81	0.70	0.81	0.75	0.81 \ 0.03	0.03	0.03
SIC 6	0.94	0.70	0.92	0.75	0.81	0.94 \ 0.09	0.09
SIC 7	0.96	0.70	0.92	0.75	0.81	0.94	0.96\ 0.14

Notes: This table presents the dependence measures implied by the estimated *skew* $t-t$ factor copula model reported in Table 9. This model implies a block equidependence structure based on the industry to which a stock belongs, and the results are presented with intra-industry dependence in the diagonal elements, and cross-industry dependence in the off-diagonal elements. The top panel present rank correlation coefficients based on 50,000 simulations from the estimated model. The bottom panel presents the theoretical upper tail depedence coefficients (upper triangle) and lower tail dependence coefficients (lower triangle) based on Propositions 2 and 3.

Table 11: Performance of methods for predicting systemic risk

	MSE		**RelMSE**	
Cut-off	*-2%*	*-4%*	*-2%*	*-4%*
	Marginal Expected Shortfall (MES)			
Brownlees-Engle	**0.9961**	**1.2023**	0.7169	0.3521
Historical	1.1479	1.6230	1.0308	0.4897
CAPM	1.1532	1.5547	0.9107	0.4623
Normal copula	1.0096	1.2521	0.6712	0.3420
t copula	1.0118	1.2580	0.6660	0.3325
Skew t copula	1.0051	1.2553	0.6030	0.3040
Skew $t-t$ factor copula	1.0012	1.2445	**0.5885**	**0.2954**
	k-Expected Shortfall (kES)			
Historical	1.1632	1.6258	1.4467	0.7653
Normal copula	1.0885	1.4855	1.3220	0.5994
t copula	1.0956	1.4921	1.4496	0.6372
Skew t copula	1.0898	1.4923	1.3370	0.5706
Skew $t-t$ factor copula	**1.0822**	**1.4850**	**1.1922**	**0.5204**

Notes: This table presents the MSE (left panel) and Relative MSE (right panel) for various methods of estimating measures of systemic risk. The top panel presents results for marginal expected shortfall (MES), defined in equation (24), and the lower panel presents results for k-expected shortfall (kES), defined in equation (25), with k set to 30. Two thresholds are considered, $C = -2\%$ and $C = -4\%$. There are 70 and 21 "event" days for MES under these two thresholds, and 116 and 36 "event" days for kES. The best-performing model for each threshold and performance metric is highlighted in bold.

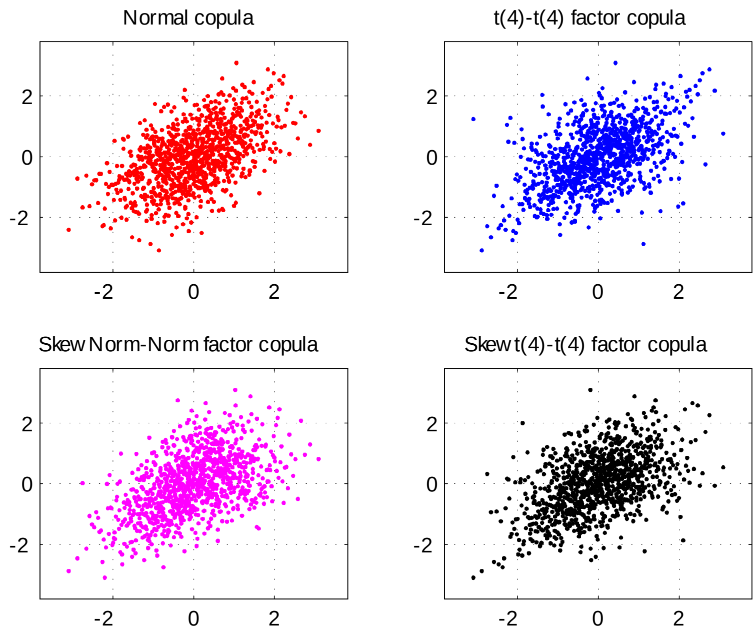

Figure 1: *Scatter plots from four bivariate distributions, all with N(0,1) margins and linear correlation of 0.5, constructed using four different factor copulas.*

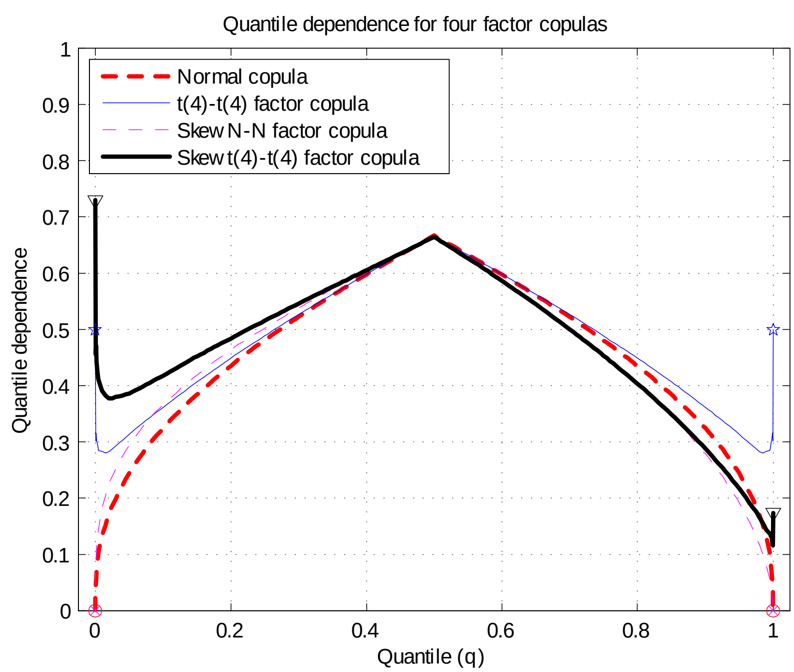

Figure 2: *Quantile dependence implied by four factor copulas, all with linear correlation of 0.5.*

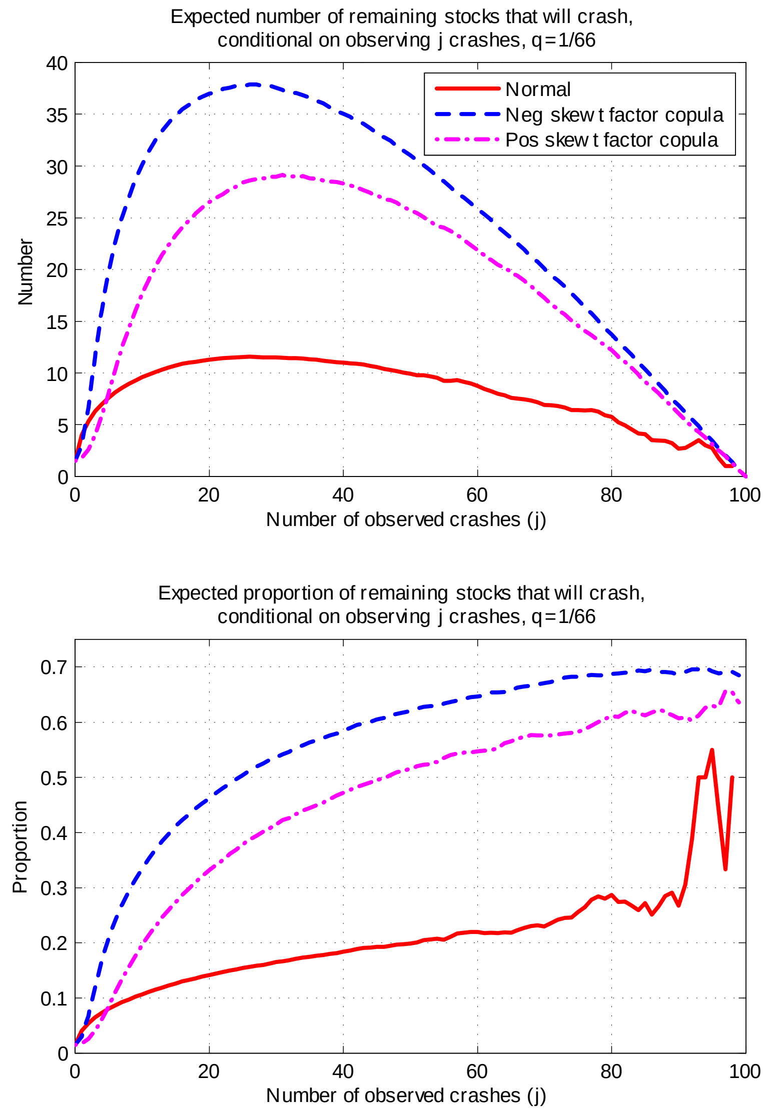

Figure 3: *Conditional on observing j out of 100 stocks crashing, this figure presents the expected number (upper panel) and proportion (lower panel) of the remaining (100-j) stocks that will crash. "Crash" events are defined as returns in the lower 1/66 tail.*

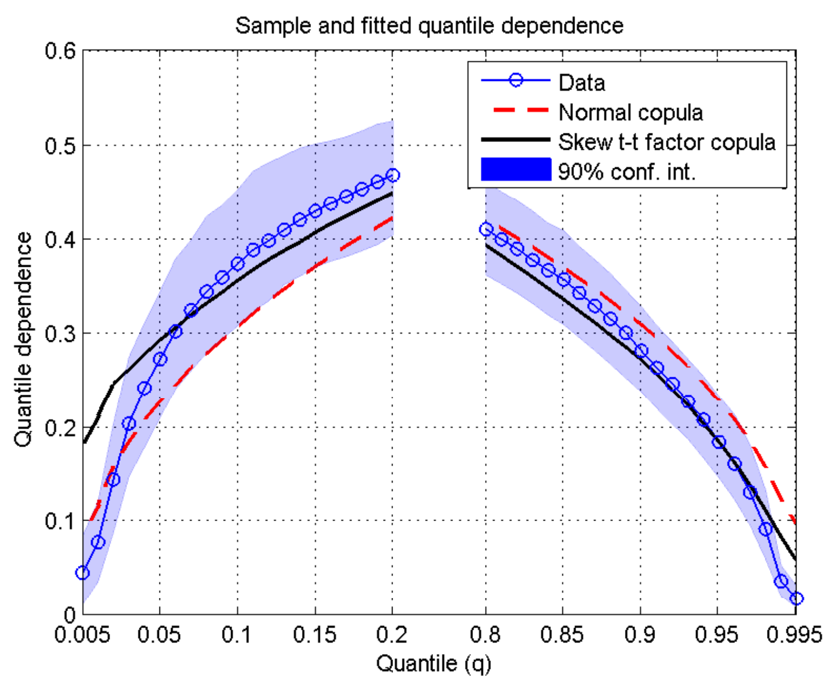

Figure 4: *Sample quantile dependence for 100 daily stock returns, along with the fitted quantile dependence from a Normal copula and from a Skew t-t factor copula, for the lower and upper tails.*

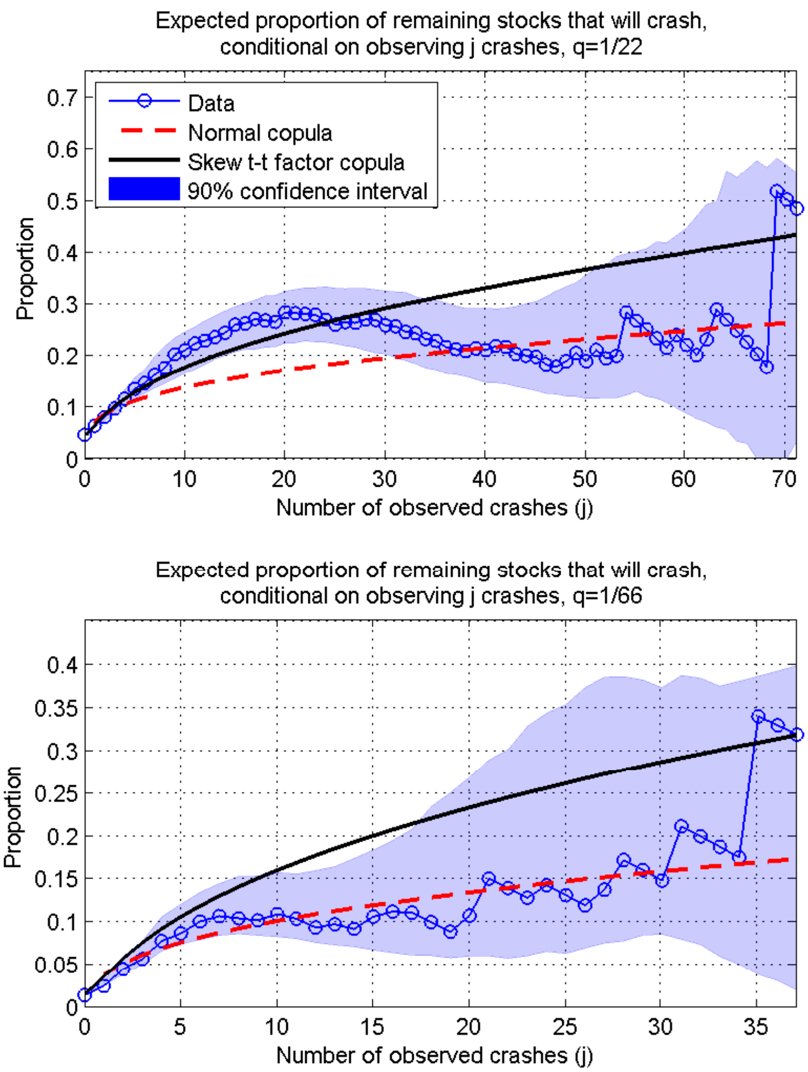

Figure 5: *Conditional on observing j out of 100 stocks crashing, this figure presents the expected proportion of the remaining (100-j) stocks that will crash. "Crash" events are defined as returns in the lower 1/22 (upper panel) and 1/66 (lower panel) tail. Note that the horizontal axes in these two panels are different, due to limited information in the joint tails.*