Time-Varying Systemic Risk: Evidence From a Dynamic Copula Model of CDS Spreads

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This article proposes a new class of copula-based dynamic models for high-dimensional conditional distributions, facilitating the estimation of a wide variety of measures of systemic risk. Our proposed models draw on successful ideas from the literature on modeling high-dimensional covariance matrices and on recent work on models for general time-varying distributions. Our use of copula-based models enables the estimation of the joint model in stages, greatly reducing the computational burden. We use the proposed new models to study a collection of daily credit default swap (CDS) spreads on 100 U.S. firms over the period 2006 to 2012. We find that while the probability of distress for individual firms has greatly reduced since the financial crisis of 2008–2009, the joint probability of distress (a measure of systemic risk) is substantially higher now than in the precrisis period. Supplementary materials for this article are available online.

KEY WORDS: Correlation; DCC; Financial crises; Tail risk.

1. INTRODUCTION

Systemic risk can be broadly defined as the risk of distress in a large number of firms or institutions. It represents an extreme event in two directions: a large loss (e.g., corresponding to a large left-tail realization for stock returns), across a large proportion of the firms. It differs from “systematic” risk in that the latter is usually taken to refer to the exposure of an asset to a general market risk, and carries no reference to the size, direction, and prevalence that is inherent in a notion of “systemic” risk. There are a variety of methods for studying risk and dependence for small collections of assets (see Patton 2013 for a recent review) but a paucity of methods for studying dependence between a large collection of assets, which is required for a general analysis of systemic risk.

Some existing methods for estimating systemic risk simplify the task by reducing the dimension of the problem to two: an individual firm and a market index. The “CoVaR” measure of Adrian and Brunnermeier (2016), for example, uses quantile regression to estimate a lower tail quantile (e.g., 0.05) of market returns conditional on a given firm having a return equal to its lower tail quantile. The “marginal expected shortfall” proposed by Brownlees and Engle (2016) estimates the expected return on a firm conditional on the market return being below some low threshold. These methods have the clear benefit of being parsimonious, but by aggregating the “nonfirm i” universe to a single market index, useful information about systemic risk may be missed. For example, firm i may impact a sizeable subset of the other firms, but fall short of precipitating a market-wide event. The objective of this article is to provide models that can be used to handle large collections of variables, and facilitate the estimation of a wider variety of systemic risk measures.

We use Sklar’s theorem (see Nelsen 2006), with an extension to conditional distributions from Patton (2006), to decompose the conditional joint distribution of a collection of N variables, $Y_t = [Y_{1t}, \ldots, Y_{Nt}]'$, into their marginal distributions and a conditional copula:

$$Y_t | F_{t-1} \sim C_t (F_{1t}, \ldots, F_{Nt}).$$

(1)

We propose new models for the time-varying conditional copula, $C_t$, that can be used to link models of the conditional marginal distributions (e.g., ARMA-GARCH models) to form a dynamic conditional joint distribution. Of central relevance to this article are cases where $N$ is relatively large, around 50 to 250. In such cases, models that have been developed for low dimension problems (say, $N < 5$) are often not applicable, either because no generalization beyond the bivariate model exists, or because such generalizations are too restrictive (e.g., Archimedean copulas have just one or two free parameters regardless of $N$, which is clearly very restrictive in high dimensions), or because the obvious generalization of the bivariate case leads to a proliferation of parameters and unmanageable computational complexity. In high-dimensional applications, the challenge is to find a balance of flexibility and parsimony.

This article makes two contributions. First, we propose a flexible and feasible model for capturing time-varying dependence in high dimensions. Our approach draws on successful ideas from the literature on dynamic modeling of high-dimensional covariance matrices and on recent work on models for general time-varying distributions. In particular, we combine the “GAS” model of Creal, Koopman, and Lucas (2011, 2013) and the factor copula model of Oh and Patton (2016) to obtain a flexible yet
parsimonious dynamic model for high-dimensional conditional distributions. This model is generally only estimable using numerical optimization techniques, and we propose a “variance targeting” (Engle and Mezrich 1996; Engle 2002) type estimator to dramatically reduce the dimension of the optimization problem. We show that this estimation method is a form of (non-linear) generalized method of moments (GMM), and thus can be treated as part of a multi-stage GMM estimation of the entire model. A realistic simulation study confirms that our proposed models and estimation methods have satisfactory properties for relevant sample sizes.

Our second contribution is a detailed study of a collection of 100 daily credit default swap (CDS) spreads on U.S. firms. The CDS market has expanded enormously over the last decade, growing 40-fold from $0.6 trillion of gross notional principal in 2001 to $25.9 trillion at the end of 2011 according to the International Swaps and Derivatives Association (ISDA), yet it has received relatively little attention, compared with equity returns, in the econometrics literature. (Interest is growing, however, see Conrad, Dittmar, and Hameed 2011; Christoffersen et al. 2013; Creal, Gramacy, and Tsay 2014; Lucas, Schwaab, and Zhang 2014; Creal and Tsay 2015 for recent work on CDS data.) We use our model of CDS spreads to provide insights into systemic risk, as CDS spreads are tightly linked to the health of the underlying firm. We find that systemic risk rose during the financial crisis, unsurprisingly. More interestingly, we also find that systemic risk remains high relative to the precrisis period, even though idiosyncratic risk has fallen.

The remainder of the article is structured as follows. Section 2 presents a dynamic copula model for high-dimensional applications, and Section 3 presents a simulation study for the proposed model and estimation method. In Section 4, we present estimation results for various models of CDS spreads. Section 5 applies the model to estimate time-varying systemic risk, and Section 6 concludes. Technical details and some additional results are presented in an online appendix.

2. A DYNAMIC COPULA MODEL FOR HIGH DIMENSIONS

In this section, we describe our approach for capturing dynamics in the dependence between a relatively large number of variables. (A review of alternative methods from the small but growing literature on this topic is presented in Section 2.5.) We consider a class of data-generating processes (DGPs) that allow for time-varying conditional marginal distributions, for example, dynamic conditional means and variances, and also possibly time-varying higher-order moments:

\[ Y_t \equiv [Y_{t1}, \ldots, Y_{Nt}] \]

where

\[ Y_{it} = \mu_{it}(\phi_{it0}) + \sigma_{it}(\phi_{it0})\eta_{it1}, \quad i = 1, 2, \ldots, N \]

(2)

\[ \eta_{i1|F_{t-1}} \sim F_{i1}(\phi_{i10}). \]

(3)

where \( \mu_{it} \) is the conditional mean of \( Y_{it} \), \( \sigma_{it} \) is the conditional standard deviation, and \( F_{i1}(\phi_{i10}) \) is a parametric distribution with zero mean and unit variance. We will denote the parameters of the marginal distributions as \( \phi \equiv [\phi^*, \ldots, \phi_{Nt}^*]' \), the parameters of the copula as \( \gamma \), and the vector of all parameters as \( \theta \equiv [\phi^*, \gamma]' \). We assume that \( F_{i1} \) is continuous and strictly increasing, which fits our empirical application, though this assumption can be relaxed. The information set is taken to be

\[ F_t = \sigma(Y_t, Y_{t-1}, \ldots). \]

Define the conditional probability integral transforms of the data as

\[ U_{it} \equiv F_{it}\left(\frac{Y_{it} - \mu_{it}(\phi_{it0})}{\sigma_{it}(\phi_{it0})}\right), \quad i = 1, 2, \ldots, N. \]

(3)

Then the conditional copula of \( Y_{i1|F_{t-1}} \) is equal to the conditional distribution of \( U_{i1|F_{t-1}} \):

\[ U_{i1|F_{t-1}} \sim C_t(\gamma_0). \]

(4)

By allowing for a time-varying conditional copula, the class of DGPs characterized by Equations (2)–(4) is a generalization of those considered by Chen and Fan (2006), for example, however the cost of this flexibility is the need to specify parametric marginal distributions. In contrast, Chen and Fan (2006), Rémillard (2010), and Oh and Patton (2013) allowed for nonparametric estimation of the marginal distributions. The parametric margin requirement arises as the asymptotic distribution theory for a model with nonparametric margins and a time-varying copula is not yet available in the literature. We attempt to mitigate this requirement in our empirical work by using flexible models for the marginal distributions, which are allowed to differ for each variable, and conducting goodness-of-fit tests to verify that they provide a satisfactory fit to the data.

2.1 Factor Copulas

In high-dimensional applications, a critical aspect of any model is imposing some form of dimension reduction. A widely used method to achieve this in economics and finance is to use a factor structure. Oh and Patton (2016) proposed using a factor model with flexible distributions to obtain a flexible class of “factor copulas.” A one-factor version of their model is the copula for the latent vector random variable \( X_t \equiv [X_{t1}, \ldots, X_{Nt}]' \) implied by the following structure:

\[ X_{it} = \lambda_{it}(\gamma_0) Z_t + \epsilon_{it}, \quad i = 1, 2, \ldots, N, \]

where

\[ Z_t \sim F_{it}(\gamma_0), \quad \epsilon_{it} \sim iid F_{it}(\gamma_0), \quad Z_t \perp \epsilon_{it}, \]

(5)

where \( F_{it}(\gamma_0) \) and \( F_{it}(\gamma_0) \) are flexible parametric univariate distributions for the common factor and the idiosyncratic variables, and \( \lambda_{it}(\gamma_0) \) is a potentially time-varying weight on the common factor. The collection of all copula parameters is \( \gamma \equiv \{\gamma_{i1}', \ldots, \gamma_{N1}'\}' \). The conditional joint distribution for \( X_t \) can be decomposed into its conditional marginal distributions and its conditional copula via Sklar’s theorem (see Nelsen 2006) for conditional distributions, see Patton (2006):

\[ X_t \sim G_t = C_t(G_{i1}(\gamma), \ldots, G_{Nt}(\gamma); \gamma). \]

(6)

Note that only the copula of these latent variables, denoted \( C_t(\gamma) \), is used as a model for the copula of the observable data \( Y_t \); the marginal distributions of \( X_t \) need not be the same as the marginal distributions of the observed data. (These marginals, \( G_{it} \), are effectively discarded.) If we impose that the marginal distributions of the observable data are also driven by the factor structure in Equation (5), then we would also use the implied marginal distributions, and this approach then becomes a standard factor model for a vector of variables. However, Oh and
Oh and Patton (2015) proposed imposing the factor structure only on the component of the multivariate model where dimension reduction is critical, namely, the copula, and allow the marginal distributions to be modeled using a potentially different approach. In this case, the factor structure in Equation (5) is used only for the copula that it implies, and this becomes a “factor copula” model.

The copula implied by Equation (5) is known in closed form for only a few particular combinations of choices of $F_z$ and $F_x$ (the most obvious example being where both of these distributions are Gaussian, in which case the implied copula is also Gaussian). For general choices of $F_z$ and $F_x$, the copula of $X$ will not be known in closed form, and thus the copula likelihood is not known in closed form. Numerical methods can be used to overcome this problem. Oh and Patton (2013) proposed simulated method of moments-type estimation of the unknown parameters, however their approach is only applicable when the conditional copula is constant. A key objective of this article is to allow the conditional copula to vary through time and so an alternate estimation approach is desirable. We use a simple numerical integration method, described in detail in the internet appendix, to overcome the lack of closed-form likelihood. This numerical integration exploits the fact that although the copula is $N$-dimensional, we need only integrate out the common factor, which is one-dimensional in the structure above.

Also, analogous to this issue in other latent factor models, the copula implied by Equation (5) is only identified up to the sign of the loadings $(\lambda_{it})$ and the sign of the common factor $(Z_t)$. That is, for a given set of loadings and common factor distribution, an identical copula can be obtained by switching the signs of all loadings and by flipping the distribution of $Z_t$ around zero. In some applications, this issue can complicate interpreting the parameters of the factor model. In our application, we exploit the fact that all of the variables are positively related, and we thus impose that the factor loadings are all positive.

Dynamics in the factor copula model in Equation (5) arise by allowing the loadings on the common factor, $\lambda_{it}$, to vary through time, and/or by allowing the distributions of the common factor and the idiosyncratic variables to change through time. For example, holding $F_z$ and $F_x$ fixed, an increase in the factor loadings corresponds to an increase in the level of overall dependence (e.g., rank correlation) between the variables. Holding the factor loadings fixed, an increase in the thickness of the tails of the distribution of the common factor increases the degree of tail dependence. In the next section, we describe how we model these dynamics.

### 2.2 Generalized Autoregressive Score Dynamics

An important feature of any dynamic model is the specification for how the parameters evolve through time. Some specifications, such as stochastic volatility models (see Shephard 2005) and related stochastic copula models (see Manner and Segers 2011; Hafner and Manner 2012) allow the varying parameters to evolve as a latent time series process. Others, such as ARCH-type models for volatility (see Engle 1982) and related models for copulas (see, e.g., Patton 2006; Jondeau and Rockinger 2006; Creal, Koopman, and Lucas 2013, e.g.) model the varying parameters as some function of lagged observables. An advantage of the latter approach over the former, especially for high-dimensional applications, is that it avoids the need to “integrate out” the innovation terms driving the latent time series processes.

Within the class of ARCH-type models (“observation driven,” in the terminology of Creal, Koopman, and Lucas 2013), the question of which function of lagged observables to use as a forcing variable in the evolution equation for the varying parameter arises. For models of the conditional variance, an obvious choice is the lagged squared residual, as in the ARCH model, but for models with parameters that lack an obvious interpretation the choice is less clear. We adopt the generalized autoregressive score (GAS) model of Creal, Koopman, and Lucas (2013) to overcome this problem. (Harvey 2013 and Harvey and Sucarrat 2014 proposed a similar method for modeling time-varying parameters, which they called a “dynamic conditional score,” or “DCS,” model.) These authors proposed using the lagged score of the density model (copula model, in our application) as the forcing variable. Specifically, for a copula with time-varying parameter $\delta_t$, governed by fixed parameter $\gamma$, we have

Let $U_{i|F_{t-1}} \sim C(\delta_t(\gamma))$

then $\delta_t = \omega + B\delta_{t-1} + As_{t-1}$

$$\nabla_{t-1} = \frac{\partial \log(c(u_{t-1};\delta_{t-1}))}{\partial \delta_{t-1}}$$

and $S_t$ is a scaling matrix, for example, the inverse Hessian or its square root. (The specification for the dynamics of the parameter $\delta_t$ can also be generalized to include observable variables, similar to the GAS model of Creal, Koopman, and Lucas (2013) to model time-varying parameters as some function of lagged observables.) If the copula likelihood is not available in closed form, as is generally the case with factor copulas, then the vector $\nabla_t$ can be obtained via numerical differentiation.

While this specification for the evolution of a time-varying parameter is clearly somewhat arbitrary, Creal, Koopman, and Lucas (2013) motivated it by showing that it nests a variety of popular and successful existing models: GARCH (Bollerslev 1986) for conditional variance; ACD (Engle and Russell 1998) for models of trade durations (the time between consecutive high-frequency observations); Davis, Dunsmuir, and Streett’s (2003) model for Poisson counts. In recent work, Blasques, Koopman, and Lucas (2014a) showed that the GAS model can be motivated information-theoretically, by showing that this approach minimizes the local Kullback–Leibler divergence between the true conditional density and the model-implied density, and Harvey (2013) motivated this specification as an approximation to a filter for a model driven by a stochastic latent parameter, or an “unobserved components” model. Finally, the recursion above can also be interpreted as the steepest ascent direction for improving the model’s fit, in terms of the likelihood, given the current value of the model parameter $\delta_t$, similar to numerical optimization algorithms such as the Gauss–Newton algorithm.
2.3 High-Dimensional Factor Copulas With GAS Dynamics

We employ the GAS model to allow for time variation in the factor loadings in the factor copula implied by Equation (5), but to keep the model parsimonious we impose that the parameters governing the “shape” of the common and idiosyncratic variables ($\gamma_c$ and $\gamma_x$) are constant. We use the skewed \( t \) distribution of Hansen (1994) as the model for \( F_t \). This distribution has two shape parameters, a degrees of freedom parameter ($\nu_t \in (2, \infty)$) and an asymmetry parameter ($\psi_t \in (-1, 1)$), and it simplifies to the standardized \( t \) distribution when $\psi = 0$. We use the standardized \( t \) distribution as the model for $F_t$ for simplicity.

The general GAS framework in Equation (7) applied to the factor loadings (\( \lambda_{it} \)) in Equation (5) would imply $N + 2N^2$ parameters governing their evolution, which represents an infeasibly large number for even moderate values of $N$. To keep the model parsimonious, we adopt a simplification from the DCC model of Engle (2002), and impose that the coefficient matrices ($B$ and $A$) are diagonal with a scalar parameter ($\beta$ and $\alpha$, respectively) on the diagonal. To avoid the estimation of an $N \times N$ scaling matrix, we set $S_t = I$. In our empirical application, below, we found slightly better results from modeling $\log \lambda_{it}$, thereby imposing positive weights, than from modeling $\lambda_{it}$ directly. We thus employ a GAS model for the log factor loadings, but the use of the logarithm here is not essential to our empirical approach:

$$
\log \lambda_{it} = \omega_i + \beta \log \lambda_{i,t-1} + \alpha s_{i,t-1} \quad i = 1, 2, \ldots, N, \quad (8)
$$

where $s_i = \log \mathbf{c}(u_i \gamma_c, \nu_t \gamma_x, \psi_t) / \log \lambda_{it}$ and $\lambda_t \equiv [\lambda_{1t}, \ldots, \lambda_{Nt}]'$.

The dynamic copula model implied by Equations (5) and (8) contains $N + 2$ parameters for the GAS dynamics, three parameters for the shape of the common and idiosyncratic variables, for a total of $N + 5$ parameters. Numerically optimizing this model when $N = 50$ or 100 represents quite a computational challenge. We propose a method to overcome this challenge by adapting an idea from the DCC model of Engle (2002), known as “variance targeting.” The nature of our GAS specification means that the variance targeting approach needs to be modified for use here.

The evolution equation for $\lambda_{it}$ in Equation (8) can be rewritten as

$$
\partial \log \lambda_{it} / \partial \log \lambda_{i,t-1} = (1 - \beta) + \alpha \partial s_{i,t-1} / \partial \log \lambda_{it} \quad (9)
$$

using the result from Creal, Koopman, and Lucas (2013) that $E_{t-1}[s_i] = 0$, and $E[\log \lambda_{it}] = \omega_i/(1 - \beta)$. The proposition below provides a method for using sample rank correlations to obtain an estimate of $E[\log \lambda_{it}]$, which eliminates the need to numerically optimize over the intercept parameters, $\omega_i$. The proposition is based on the following assumption.

**Assumption 1.**

(a) The conditional copula of $Y_t | \mathcal{F}_{t-1}$ is the time-varying factor copula given in Equations (5) and (8).

(b) The process $\{\lambda_t\}$ generated by Equation (8) is strictly stationary.

(c) Let $\rho_{iX} \equiv \text{vech}(\text{RankCorr}_{t-1}[X_i])$. Then $\log \lambda_t$ is a linear function of $\rho_{iX}$.

(d) Let $\rho_{iX} \equiv \text{RankCorr}_{t-1}[X_i, X_j]$ and $\rho^{ij}_{tX} \equiv \text{Corr}[X_i, X_j]$. Then, for fixed values of $(\gamma_c, \gamma_x)$, the mapping $\rho_{iX} \equiv \psi(\rho^{ij}_{tX})$ is strictly increasing.

Part (a) of this assumption makes explicit that the copula of the data is the GAS-factor copula model, and so the conditional copula of $Y_t | \mathcal{F}_{t-1}$ is the same as that of $X_t | \mathcal{F}_{t-1}$. Blasques, Koopman, and Lucas (2014b) provided conditions under which univariate GAS models satisfy stationarity conditions, and Blasques, Koopman, and Lucas (2014a) provided results for GAS models with a vector-dependent variable but a scalar time-varying parameter; corresponding theoretical results for our multivariate case are not yet available in the literature, and in part (b) we simply assume that stationarity holds. Part (c) formalizes the applicability of a Taylor series expansion of the function mapping $\rho_t$ to $\lambda_t$. In practice, this assumption will hold only approximately, and its applicability needs to be verified via simulation, which we discuss further in Section 3 and in the internet appendix. Part (d) enables us to map rank correlations to linear correlations. Note that we can take $(\gamma_c, \gamma_x)$ as fixed, as we call this mapping for each evaluation of the log-likelihood. Importantly, this mapping can be computed prior to estimation, and then just called during estimation, rather than recomputed each time the likelihood function is evaluated, which greatly speeds up estimation.

**Proposition 1.** Let Assumption 1 hold, and denote the vech of the rank correlation matrix of the standardized residuals, $\eta_t$, as $\hat{\rho}_t$ and its sample analog as $\hat{\rho}_t$. Then:

(i) $E[\log \lambda_t] = H(\hat{\rho}_t)$, where $H$ is defined in Equation (A.4).

(ii) $\hat{\lambda} = H(\hat{\rho}_t)$ is a GMM estimator of $E[\log \lambda_t]$.

The proof of this proposition is in the appendix. Part (i) of the above proposition provides the mapping from the population rank correlation of the standardized residuals to the mean of the (log) factor loadings, which is the basis for considering a variance-targeting type estimator. Part (ii) shows that the sample analog of this mapping can be interpreted as a standard GMM estimator. This is useful as it enables us to treat the estimation of the complete vector of parameters as multi-stage GMM. Under standard regularity conditions, this can be shown to yield consistent and asymptotically normal parameter estimates (see White 1994; Engle and Sheppard 2001, and in particular Newey and McFadden 1994, Theorem 6.1). That is, if we let $\hat{\theta}_{\text{MS-GMM}} \equiv [\hat{\theta}_{\text{MS-GMM}}, \hat{\gamma}_{\text{MS-GMM}}]'$ denote the collection of all estimated parameters, then

$$
\sqrt{T}(\hat{\theta}_{\text{MS-GMM}} - \theta^*) \overset{d}{\Rightarrow} N(0, V_{\text{MS-GMM}}^*) \quad (10)
$$

Consistent estimation of $V_{\text{MS-GMM}}^*$ is theoretically possible, however in high dimensions it is not computationally feasible. Gonçalves et al. (2013) provided conditions under which a block bootstrap may be used to obtain valid standard errors on parameters estimated via multi-stage GMM. The resulting standard errors are not higher-order efficient, like some bootstrap inference methods, but they do enable us to avoid having to handle Hessian matrices of size on the order of $N \times N$. Note that sample rank correlations cannot in general be considered as moment-based estimators, as they depend on the sample ranks.
of the observed data, and studying their estimation properties requires alternative techniques. However, we exploit the fact that the marginal distributions of the data are known up to an unknown parameter vector, and thus rank correlation can be obtained as a sample moment of a nonlinear function of the data.

2.4 Equidependence Versus Heterogenous Dependence

To investigate whether we can further reduce the number of free parameters in this model, we consider two restrictions of the model in Equation (8), motivated by the “dynamic equicorrelation” model of Engle and Kelly (2012). If we impose that \( \omega_i = \omega \forall i \), then the pairwise dependence between each of the variables will be identical, leading to a “dynamic equidependence” model. (The copula implied by this specification is “exchangeable” in the terminology of the copula literature.) In this case, we have only six parameters to estimate, independent of the number of variables \( N \), vastly reducing the estimation burden, but imposing a lot of homogeneity on the model.

An intermediate step between the fully flexible model and the equidependence model is to group the assets using some ex ante information (e.g., by industry) and impose homogeneity only within groups. This leads to a “block equidependence” copula model, with

\[
X_{it} = \lambda_{g(i),t} Z_i + \varepsilon_{it}, \quad i = 1, 2, \ldots, N
\]

\[
\log \lambda_{g,t} = \alpha_g + \beta \log \lambda_{g,t-1} + \alpha \lambda_{g,t-1}, \quad g = 1, 2, \ldots, G
\]

(11)

where \( g(i) \) is the group to which variable \( i \) belongs, and \( G \) is the number of groups. In this case, the number of parameters to estimate in the copula model is \( G + 5 \). In our empirical application, we have \( N = 100 \) and we consider grouping variables into \( G = 5 \) industries, meaning this model has 10 parameters to estimate rather than 105. In our empirical analysis below, we compare these two restricted models (\( G = 1 \) and \( G = 5 \)) with the “heterogenous dependence” model that allows a different factor for each variable (and so \( G = N \)).

2.5 Other Models for Dynamic, High-Dimensional Copulas

As noted above, the literature contains relatively few models for dynamic, high-dimensional, copulas. Exceptions to this are discussed here. Lucas, Schwaab, and Zhang (2014) combined GAS dynamics with a skewed \( t \) copula to model 10 sovereign CDS spreads. A similar model, though with an alternative skew \( t \) specification and with Engle’s (2002) DCC dynamics, is used by Christoffersen et al. (2012, 2013). The former of these two articles analyzes equity returns on up to 33 national stock indices, while the latter studies weekly equity returns and CDS spreads on 233 North American firms (and is the largest time-varying copula model in the extant literature). Creal and Tsay (2015) proposed a stochastic copula model based on a factor structure, and used Bayesian estimation methods to apply it to an unbalanced panel of CDS spreads and equity returns on 100 firms. Almeida, Czado, and Manner (2012) used “vine” copulas to model the dependence between 30 German stock return series, with dynamics captured through a stochastic volatility-type equation for the parameters of the copula. Stöber and Czado (2012) also used vine copulas, combined with a regime-switching model for dynamics, to model dependence between 10 German stock returns.

3. SIMULATION STUDY

This section presents an analysis of the finite sample properties of maximum likelihood estimation for factor copulas with GAS dynamics. Factor copulas do not have a closed-form likelihood, and we approximate the likelihood using standard numerical integration methods, details of which can be found in the internet appendix.

We consider three different copula models for the Monte Carlo simulation: a dynamic equidependence model (\( G = 1 \)), a dynamic block equidependence model (\( G = 10 \)), and a dynamic heterogenous dependence model (\( G = N \)), all of them governed by

\[
X_{it} = \lambda_{g(i),t} Z_i + \varepsilon_{it}, \quad i = 1, 2, \ldots, N
\]

\[
\log \lambda_{g,t} = \alpha_g + \beta \log \lambda_{g,t-1} + \alpha \lambda_{g,t-1}, \quad g = 1, 2, \ldots, G
\]

\[
Z \sim Skew \ t (v, \psi_i)
\]

\[
\varepsilon_i \sim iid \ t(v_i), \quad \varepsilon_i \perp \perp Z \forall i
\]

(12)

We set \( N = 100 \) to match the number of series in our empirical application below. For simplicity, we impose that \( v_2 = v, \psi \), and we estimate \( v^{-1} \) rather than \( v \), so that Normality is nested at \( v^{-1} = 0 \) rather than \( v \rightarrow \infty \). Broadly matching the parameter estimates we obtain in our empirical application, we set \( \omega = 0, \beta = 0.98, \alpha = 0.05, v = 5, \) and \( \psi = 0.1 \) for the equidependence model. The block equidependence model uses the same parameters but sets \( \omega_1 = -0.03 \) and \( \omega_{10} = 0.03 \), and with \( \omega_2 \) to \( \omega_5 \) evenly spaced between these two bounds, and the heterogenous dependence model similarly uses \( \omega_1 = -0.03 \) and \( \omega_{100} = 0.03 \), with \( \omega_2 \) to \( \omega_{99} \) evenly spaced between these two bounds. Rank correlations implied by these values range from 0.1 to 0.7. With these choices of parameter values and dependence designs, various dynamic dependence structures are covered, and asymmetric tail dependence, which is a common feature of financial data, is also allowed. We use a sample size of \( T = 500 \) and we repeat each simulation 100 times.

The results for the equidependence model presented in Panel A of Table 1 reveal that the average estimated bias for all parameters is small, and the estimates are centered on true values. The results for the block equidependence model, presented in Panel B, are also satisfactory, and, as expected, the estimation error in the parameters is generally slightly higher for this more complicated model.

The heterogenous dependence model is estimated using the variance targeting-type approach for the intercepts, \( \omega_i \), described in Section 2.3, combined with numerical optimization for the remaining parameters. The internet appendix presents simulation results that verify the applicability of Assumption 1 for this model, and complete simulation results for this model are presented in Table A1 of the internet appendix. This table confirms that the approach leads to estimators with satisfactory
finite-sample properties. The standard errors on the estimated intercept parameters are approximately twice as large, on average, as in the block equidependence case, however this model has seven times as many parameters as the block equidependence (104 vs. 14) and so some loss in accuracy is inevitable. Importantly, all estimated parameters are approximately centered on their true values, confirming that the assumptions underlying Proposition 1 are applicable for this model.

4. DATA DESCRIPTION AND ESTIMATION RESULTS

4.1 CDS Spreads

We apply the new dynamic copula model proposed in the previous section to daily credit default swap (CDS) spreads, obtained from Markit. In brief, a CDS is a contract in which the seller provides insurance to the buyer against any losses resulting from a default by the “reference entity” within some horizon. We focus on North American corporate CDS contracts, and the reference entities are thus North American firms. The CDS spread, usually measured in basis points and payable quarterly by the buyer to the seller, is the cost of this insurance. See Duffie and Singleton (2003) and Hull (2012) for more detailed discussions of CDS contracts, and see Barclays “CDS Handbook” (2010) for institutional details.

A key reason for interest in CDS contracts is the sensitivity of CDS spreads to changes in market perceptions of the probability of default, see Conrad, Dittmar, and Hameed (2011), Creal, Gramacy, and Tsay (2014), Christoffersen et al. (2013), and Liu (2014) for recent empirical studies of implied default probabilities. Under some simplifying assumptions (such as a constant risk free rate and default hazard rate) see Carr and Wu (2011), for example, it can be shown that the CDS spread in basis points is

$$S_{it} = 100^2 P_{it}^Q L_{it}, \quad (13)$$

where $L_{it}$ is the loss given default (sometimes shortened to “LGD,” and often assumed to equal 0.6 for U.S. firms) and $P_{it}^Q$ is the implied probability of default. The same formula can also be obtained as a first-order approximation at $P_{it}^Q \approx 0$ for other more complicated pricing equations. This expression can be written in terms of the objective probability of default, $P_{it}^O$:

$$S_{it} = 100^2 P_{it}^O \mathcal{M}_{it} L_{it}, \quad (14)$$

where $\mathcal{M}_{it}$ is the market price of risk (stochastic discount factor). An increase in a CDS spread can be driven by an increase in the LGD, an increase in the market price of default risk for this firm, or an increase in the objective probability of default. Any one of these three effects is indicative of a worsening of the health of the underlying firm.

In the analysis below, we work with the log-difference of CDS spreads, to mitigate their autoregressive persistence, and under this transformation we obtain:

$$Y_{it} \equiv \Delta \log S_{it} = \Delta \log P_{it}^O + \Delta \log \mathcal{M}_{it} + \Delta \log L_{it}. \quad (15)$$

If the loss given default is constant then the third term above vanishes, and if we assume that the market price of risk is constant (as in traditional asset pricing models) or evolves slowly (e.g., with a business cycle-type frequency) then daily changes in CDS spreads can be attributed primarily to changes in the objective probability of default. We will use this to guide our interpretation of the empirical results below, but we emphasize here that an increase in any of these three terms represents “bad

<table>
<thead>
<tr>
<th>Variable</th>
<th>True</th>
<th>Bias</th>
<th>Std</th>
<th>Median</th>
<th>90%</th>
<th>10%</th>
<th>Diff (90%–10%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$</td>
<td>0.000</td>
<td>0.005</td>
<td>0.015</td>
<td>0.001</td>
<td>0.027</td>
<td>−0.003</td>
<td>0.030</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.050</td>
<td>0.000</td>
<td>0.003</td>
<td>0.050</td>
<td>0.051</td>
<td>0.048</td>
<td>0.003</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.980</td>
<td>0.002</td>
<td>0.004</td>
<td>0.980</td>
<td>0.989</td>
<td>0.979</td>
<td>0.010</td>
</tr>
<tr>
<td>$\nu^{-1}$</td>
<td>0.200</td>
<td>0.001</td>
<td>0.006</td>
<td>0.200</td>
<td>0.206</td>
<td>0.195</td>
<td>0.010</td>
</tr>
<tr>
<td>$\psi_{z}$</td>
<td>0.100</td>
<td>0.005</td>
<td>0.017</td>
<td>0.100</td>
<td>0.118</td>
<td>0.097</td>
<td>0.021</td>
</tr>
</tbody>
</table>

Panel A: Equidependence

Panel B: Block equidependence

NOTES: This table presents results from the simulation study described in Section 3. The online appendix contains results for the “heterogenous dependence” model.
news” for firm i, and so the isolation of the objective probability of default is not required for our interpretations to follow.

4.2 Summary Statistics

Our sample period spans January 2006 to April 2012, a total of 1644 days. We study the 5-year CDS contract, which is the most liquid horizon (see Barclays Capital 2010), and we use “XR” (“no restructuring”) CDS contracts, which became the convention for North America following the CDS market standardization in 2009 (the so-called “Big Bang”). To obtain a set of active, economically interesting, CDS data, we took all 125 individual firms in the CDS index covering our sample period (CDX NA IG Series 17). Of these, 90 firms had data that covered our entire sample period, and 10 firms had no more than three missing observations. We use these 100 firms for our analysis. (Of the remaining 25 firms, 6 are not U.S.-based firms and 1 firm stopped trading because of a firm split. None of the firms defaulted over this sample period.) We note here that Markit selects CDS contracts for inclusion in CDX indices based on their liquidity (Markit Group 2015) not on their broader “representativeness,” in some sense, and so the results presented below are representative of those for liquid CDS contracts, not necessarily for all CDS contracts, or all firms.

A plot of the CDS spreads used in our analysis is presented in Figure 1, which reveals that the average CDS spread was around 100 basis points (bps), and it varied from a low (averaged across firms) of 24 bps on February 22, 2007, to a high of 304 bps on March 9, 2009.

The levels of our CDS spread data are suggestive of a large autoregressive root, with the median first-order autocorrelation across all 100 series being 0.996 (the minimum is 0.990). Further, augmented Dickey–Fuller tests reject the null hypothesis of a unit root at the 0.05 level for only 12 series. Like interest rate time series, these series are unlikely to literally obey a random walk, as they are bounded below, however we model all series in log differences to avoid the need to consider these series as near unit root processes.

Table 2 presents summary statistics on our log-differenced data. Of particular note is the positive skewness of the log-differences in CDS spreads (average skewness is 1.087, and skewness is positive for 89 out of 100 series) and the excess kurtosis (25.531 on average, and greater than 3 for all 100 firms). Ljung–Box tests for autocorrelation at up to the tenth lag find significant (at the 0.05 level) autocorrelation in 98 out of 100 series (across firms) percent change in CDS spreads for the same time period. We use the skewed t distribution of

4.3 Conditional Mean and Variance Models

Daily log-differences of CDS spreads have more autocorrelation than is commonly found for daily stock returns (e.g., the average first-order autocorrelation is 0.161) and so the model for the conditional mean of our data needs more structure than the commonly used constant model for daily stock returns. We use an AR(5) model, and augment it with a lag of the market variable (an equal-weighted average of all 100 series) to capture any dynamic spillover effects. We show below that this model passes standard specification tests:

\[ Y_{it} = \phi_0 + \sum_{j=1}^{5} \phi_{j,i} Y_{i,t-j} + \phi_{m,i} Y_{m,t-1} + e_{it}. \]  

For the market return, we use the same model (omitting, of course, a repeat of the first lag of the market return). We need a model for the market return as we use the residuals from the market return model in our conditional variance specification.

Our model for the conditional variance is the asymmetric volatility model of Glosten, Jagannathan, and Runkle (1993), the “GJR-GARCH” model. The motivation for the asymmetry in this model is that “bad news” about a firm increases its future volatility more than good news. For stock returns, bad news comes in the form of a negative residual. For CDS spreads, on the other hand, bad news is a positive residual, and so we reverse the direction of the indicator variable in the GJR-GARCH model to reflect this. In addition to the standard GJR-GARCH terms, we also include terms relating to the lagged market residual:

\[ V_{t-1} [e_{it}] \equiv \sigma_i^2 \equiv \omega_i + \beta_i \sigma_i^2_{t-1} + \alpha_i e_{i,t-1}^2 + \delta_i e_{i,t-1}^2 1|e_{i,t-1} > 0] \]
\[ + \alpha_m e_{m,t-1}^2 + \delta_m e_{m,t-1}^2 1|e_{m,t-1} > 0). \]  

Finally, we specify a model for the marginal distribution of the standardized residuals, \( \eta_{it} \). We use the skewed t distribution of
Table 2. Summary statistics for log-differences of daily CDS spreads

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>5%</th>
<th>25%</th>
<th>Median</th>
<th>75%</th>
<th>95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>5.589</td>
<td>−1.634</td>
<td>2.559</td>
<td>5.529</td>
<td>8.521</td>
<td>13.817</td>
</tr>
<tr>
<td>Std dev</td>
<td>378.892</td>
<td>308.636</td>
<td>347.627</td>
<td>373.460</td>
<td>400.385</td>
<td>476.533</td>
</tr>
<tr>
<td>1st-order autocorr</td>
<td>0.161</td>
<td>0.030</td>
<td>0.121</td>
<td>0.164</td>
<td>0.217</td>
<td>0.267</td>
</tr>
<tr>
<td>Skewness</td>
<td>1.087</td>
<td>−0.285</td>
<td>0.354</td>
<td>0.758</td>
<td>1.488</td>
<td>3.629</td>
</tr>
<tr>
<td>5%</td>
<td>−514.574</td>
<td>−622.282</td>
<td>−551.334</td>
<td>−509.554</td>
<td>−474.027</td>
<td>−415.651</td>
</tr>
<tr>
<td>25%</td>
<td>−144.195</td>
<td>−172.319</td>
<td>−155.635</td>
<td>−145.415</td>
<td>−134.820</td>
<td>−111.993</td>
</tr>
<tr>
<td>Median</td>
<td>−2.324</td>
<td>−9.045</td>
<td>−3.644</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>75%</td>
<td>132.127</td>
<td>95.168</td>
<td>120.514</td>
<td>131.019</td>
<td>144.363</td>
<td>174.645</td>
</tr>
<tr>
<td>95%</td>
<td>570.510</td>
<td>457.775</td>
<td>537.093</td>
<td>568.331</td>
<td>612.769</td>
<td>684.984</td>
</tr>
</tbody>
</table>

NOTES: This table presents summary statistics across the 100 marginal distributions of log-differences of daily CDS spreads measured in basis points. The columns present the mean and quantiles from the cross-sectional distribution of the measures listed in the rows.

Hansen (1994), which allows for nonzero skewness and excess kurtosis:

\[ \eta_{it} \equiv \frac{\varepsilon_{it}}{\sigma_{it}} \sim \text{iid Skew } t(\nu_i, \psi_i). \]  

Table 3 summarizes the results of estimating the above models on the 100 time series. For the conditional mean model, we find strong significance of the first three AR lags, as well as the lagged market return. The conditional variance models reveal only mild statistical evidence of asymmetry in volatility, however the point estimates suggest that “bad news” (a positive residual) increases future volatility about 50% more than good news. The average estimated degrees of freedom parameter is 3.6, suggestive of fat tails, and the estimated skewness parameter is positive for 94 firms, and significantly different from zero for 41 of these, indicating positive skewness.

We now discuss goodness-of-fit tests for the marginal distribution specifications. We first use the Ljung–Box test to check the adequacy of these models for the conditional mean and variance, and we are able to reject the null of zero autocorrelation up to the tenth lag for only nine of the residual series, and only two of the squared standardized residual series. We conclude that these models provide a satisfactory fit to the conditional means and variances of these series. Next, we use the Kolmogorov–Smirnov test to investigate the fit of the skewed \( t \) distribution for the standardized residuals, using 100 simulations to obtain critical values that capture the parameter estimation error, and we reject the null of correct specification for 11 of the 100 firms.

Table 3. Marginal distribution parameter estimates

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>5%</th>
<th>25%</th>
<th>Median</th>
<th>75%</th>
<th>95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi_{0,i} )</td>
<td>3.029</td>
<td>−3.760</td>
<td>0.247</td>
<td>3.116</td>
<td>5.861</td>
<td>10.165</td>
</tr>
<tr>
<td>( \phi_{1,i} )</td>
<td>0.005</td>
<td>−0.179</td>
<td>−0.062</td>
<td>0.010</td>
<td>0.082</td>
<td>0.153</td>
</tr>
<tr>
<td>( \phi_{2,i} )</td>
<td>0.025</td>
<td>−0.039</td>
<td>−0.001</td>
<td>0.025</td>
<td>0.050</td>
<td>0.084</td>
</tr>
<tr>
<td>( \phi_{3,i} )</td>
<td>−0.002</td>
<td>−0.058</td>
<td>−0.028</td>
<td>−0.004</td>
<td>0.021</td>
<td>0.064</td>
</tr>
<tr>
<td>( \phi_{4,i} )</td>
<td>0.006</td>
<td>−0.046</td>
<td>−0.014</td>
<td>0.006</td>
<td>0.033</td>
<td>0.054</td>
</tr>
<tr>
<td>( \phi_{5,i} )</td>
<td>0.004</td>
<td>−0.055</td>
<td>−0.022</td>
<td>0.005</td>
<td>0.027</td>
<td>0.060</td>
</tr>
<tr>
<td>( \phi_{m,i} )</td>
<td>0.387</td>
<td>0.163</td>
<td>0.303</td>
<td>0.372</td>
<td>0.480</td>
<td>0.638</td>
</tr>
<tr>
<td>( \omega_i \times 10^4 )</td>
<td>5.631</td>
<td>1.401</td>
<td>3.111</td>
<td>5.041</td>
<td>7.260</td>
<td>13.381</td>
</tr>
<tr>
<td>( \beta_i )</td>
<td>0.741</td>
<td>0.595</td>
<td>0.699</td>
<td>0.746</td>
<td>0.794</td>
<td>0.845</td>
</tr>
<tr>
<td>( \alpha_i )</td>
<td>0.114</td>
<td>0.052</td>
<td>0.087</td>
<td>0.106</td>
<td>0.141</td>
<td>0.181</td>
</tr>
<tr>
<td>( \delta_i )</td>
<td>0.022</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.042</td>
<td>0.086</td>
</tr>
<tr>
<td>( \alpha_{m,i} )</td>
<td>0.223</td>
<td>0.037</td>
<td>0.137</td>
<td>0.206</td>
<td>0.297</td>
<td>0.494</td>
</tr>
<tr>
<td>( \delta_{m,i} )</td>
<td>0.072</td>
<td>0.000</td>
<td>0.000</td>
<td>0.059</td>
<td>0.114</td>
<td>0.233</td>
</tr>
<tr>
<td>( \nu_i )</td>
<td>3.620</td>
<td>2.877</td>
<td>3.293</td>
<td>3.571</td>
<td>3.921</td>
<td>4.496</td>
</tr>
<tr>
<td>( \psi_i )</td>
<td>0.043</td>
<td>−0.003</td>
<td>0.024</td>
<td>0.042</td>
<td>0.062</td>
<td>0.089</td>
</tr>
</tbody>
</table>

NOTES: The table presents summaries of the estimated AR(5)-GJR-GARCH(1,1)-Skew \( t \) marginal distribution models (Equations (16)–(18)) estimated on log-difference of daily CDS spreads. The columns present the mean and quantiles from the cross-sectional distribution of the parameters listed in the rows. The bottom panel shows the number of rejections (at the 0.05 level) across 100 firms from Ljung–Box tests for serial correlation up to 10 lags. The first row is for standardized residuals of log-difference of daily CDS spreads and the second row for squared standardized residuals. The bottom panel shows the number of rejections across 100 firms from the Kolmogorov–Smirnov test of the Skew \( t \) distribution used for the standardized residuals.
This is slightly higher than the level of the test (0.05), but we do not pursue the use of a more complicated marginal distribution model for those 11 firms in the interests of parsimony and comparability.

4.4 The CDS “Big Bang”

On April 8, 2009, the CDS market underwent changes driven by a move toward more standardized CDS contracts. Details of these changes are described in Markit Group (2009). It is plausible that the changes to the CDS market around this so-called “Big Bang” changed the dynamics and distributional features of CDS time series, and we test for that possibility here. We do so by allowing the parameters of the mean, variance, and marginal distribution models to change on the date of the Big Bang, and we test the significance of these changes. We have 591 prebreak observations and 1053 post-break observations. (It is worth noting that the turmoil in financial markets in 2007–09 makes identifying any single event as a source of a structural break almost impossible. We have good reason to suspect that the Big Bang led to a structural break, but it is possible that a better break date, in terms of model fit, is earlier or later than the date we use.)

We find that the conditional mean parameters changed significantly (at the 0.05 level) for 39 firms, and the conditional variance and marginal density shape parameters changed significantly for 66 firms. In what follows, the results we report are based on models that allow for a structural break in the mean, variance, and distribution parameters. Given the prevalence of these changes, all of the copula models we consider allow for a break at the date of the Big Bang.

4.5 Comparing Models for the Conditional Copula

The class of high-dimensional dynamic copula models described in Section 2 includes a variety of possible specifications: static versus GAS dynamics; normal versus skew t-t factor copulas; equidependence versus block copula dependence versus heterogenous dependence.

Table 4 presents results for six different dynamic models (a corresponding table for the six static copula models is available in the internet appendix). Bootstrap standard errors are presented in parentheses below the estimated parameters. (We use the stationary block bootstrap of Politis and Romano (1994) with an average block length of 120 days, applied to the log-difference of the CDS spreads, and we use 100 bootstrap replications.) Similar to other applications of GAS models (see Creal, Koopman, and Lucas 2011, 2013), we find strong persistence, with the \( \beta \) parameter ranging from 0.85 to 0.99. (Note that the \( \beta \) parameter in GAS models plays the same role as \( \alpha + \beta \) in a GARCH(1,1) model, see Example 1 in Creal, Koopman, and Lucas 2013.) We also find that the inverse degrees of freedom parameters are greater than zero (i.e., the factor copula is not Normal), which we test formally below. For the two more parsimonious models, we do not find the common factor to be significantly asymmetrically distributed, but for our preferred model based on heterogenous dependence, we find that the asymmetry parameter for the common factor is positive, and significantly so on heterogenous dependence, we find that the asymmetry parameter for the common factor is positive, and significantly so.

Table 4 shows that the estimated degrees of freedom parameter for the common factor is larger than that for the idiosyncratic term. Oh and Patton (2016) showed that when these two parameters differ the tail dependence implied by this factor copula is on the boundary: either zero (\( \nu_c > \nu_z \)) or one (\( \nu_c < \nu_z \)); only when these parameters are equal can tail dependence lie inside (0, 1). We test the significance of the
The difference between these two parameters by estimating a model with them imposed to be equal and then conducting a likelihood ratio test, the log-likelihoods from these two models are compared with them imposed to be equal and then conducting a likelihood ratio test. The difference between these two parameters by estimating a model with them imposed to be equal and then conducting a likelihood ratio test, the log-likelihoods from these two models are compared.

Table 5. Model comparison results

<table>
<thead>
<tr>
<th></th>
<th>Static</th>
<th>GAS</th>
<th>Factor (v, ψ, v)</th>
<th>Static</th>
<th>GAS</th>
<th>Factor (v, ψ, v, v)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equidependence</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>No. of param</td>
<td>2</td>
<td>6</td>
<td>6</td>
<td>10</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>log L</td>
<td>36,185</td>
<td>38,395</td>
<td>39,223</td>
<td>40,688</td>
<td>39,508</td>
<td>40,983</td>
</tr>
<tr>
<td>AIC</td>
<td>-72,366</td>
<td>-76,778</td>
<td>-78,434</td>
<td>-81,356</td>
<td>-79,000</td>
<td>-81,942</td>
</tr>
<tr>
<td>BIC</td>
<td>-72,364</td>
<td>-76,791</td>
<td>-78,427</td>
<td>-81,344</td>
<td>-79,990</td>
<td>-81,927</td>
</tr>
<tr>
<td>Block equidependence</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>No. of param</td>
<td>10</td>
<td>14</td>
<td>14</td>
<td>18</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>log L</td>
<td>36,477</td>
<td>38,518</td>
<td>39,441</td>
<td>40,799</td>
<td>39,757</td>
<td>41,165</td>
</tr>
<tr>
<td>AIC</td>
<td>-72,934</td>
<td>-77,008</td>
<td>-78,854</td>
<td>-81,562</td>
<td>-79,482</td>
<td>-82,290</td>
</tr>
<tr>
<td>BIC</td>
<td>-72,922</td>
<td>-76,991</td>
<td>-78,837</td>
<td>-81,540</td>
<td>-79,463</td>
<td>-82,266</td>
</tr>
<tr>
<td>Heterogenous dependence</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>No. of param</td>
<td>200</td>
<td>204</td>
<td>204</td>
<td>208</td>
<td>210</td>
<td></td>
</tr>
<tr>
<td>log L</td>
<td>37,652</td>
<td>39,361</td>
<td>40,357</td>
<td>41,522</td>
<td>40,628</td>
<td>41,913</td>
</tr>
<tr>
<td>AIC</td>
<td>-74,904</td>
<td>-78,314</td>
<td>-80,036</td>
<td>-82,628</td>
<td>-80,844</td>
<td>-83,406</td>
</tr>
<tr>
<td>BIC</td>
<td>-74,661</td>
<td>-78,066</td>
<td>-80,058</td>
<td>-82,375</td>
<td>-80,594</td>
<td>-83,151</td>
</tr>
</tbody>
</table>

NOTES: This table presents the log-likelihood at the estimated parameters, as well as the Akaike and Bayesian Information criteria, for a variety of copula models. The preferred model according to each of these criteria is highlighted in bold. Also presented is the number of estimated parameters; note that this accounts for the fact that we allow for a structural break on the common factor for each series (the “heterogenous dependence” model) and allowing for dynamics using the GAS structure described in the previous section.

Figure 2 presents the time-varying factor loadings implied by this model, summarized by averaging the loadings across all firms in the same industry. Figure A2 in the internet appendix presents corresponding averages of all pair-wise correlations between firms in the same pair of industries. (Thus, the plotted factor loadings and rank correlations are smoother than any individual rank correlation plot.) Also presented in Figure A2 are 60-day rolling window rank correlations, again averaged across pairs of the firms in the same pair of industries. This figure reveals a secular increase in the correlation between CDS spreads, rising from around 0.1 in 2006 to around 0.5 in 2013. Interestingly, rank correlations do not appear to spike during the financial crisis, unlike individual volatilities and probabilities of default; rather they continue a mostly steady rise through the sample period.
5. TIME-VARYING SYSTEMIC RISK

In this section, we use the dynamic multivariate model presented above to obtain estimates of measures of systemic risk. In many recent studies, systemic risk is taken to refer to breakdowns of the financial system, but in this article we use a broader definition in which systemic risks can afflict any given system, including those that include both financial and nonfinancial firms. Also note that while the factor structure employed in the copula model in Equations (5) and (6) is similar to (and, indeed, in part inspired by) common models for capturing systematic risk, we use our model to study systemic risk by looking at its implications for measures of the probability of large crashes across a large proportion of the firms under analysis.

A variety of measures of systemic risk have been proposed in the literature to date. One influential measure is “CoVaR,” proposed by Adrian and Brunnermeier (2016), which uses quantile regression to estimate the lower tail (e.g., 0.05) quantile of market returns conditional on a given firm having a return equal to its lower tail quantile. This measure provides an estimate of how firm-level stress spills over to the market index. An alternative measure is “marginal expected shortfall” (MES) proposed by Brownlees and Engle (2016), which estimates the expected return on a firm conditional on the market return being below some low threshold. Segoviano and Goodhart (2009) and Giesecke and Kim (2011) proposed measuring systemic risk via the probability that a “large” number of firms are in distress. Lucas, Schwaab, and Zhang (2014) used the same measure applied to European sovereign debt. Hartmann, Straetmans, and de Vries (2006) considered a measure based on the probability of a large number of firms being in distress conditional on a subset of firms being in distress. Huang, Zhou, and Zhu (2009) suggested using the price of a hypothetical contract insuring against system-wide distress, valued using a mix of CDS and equity data, as a measure of systemic risk. Schwaab (2010) and Bisias et al. (2012) presented reviews of these and other measures of systemic risk. Giglio, Kelly, and Pruitt (2016) considered a large collection of systemic risk measures, and showed that an index based on the JPD is only estimable via simulations from our model, and we obtain these using 10,000 simulations. Given the computational burden, we compute estimates only every 20 trading days (approximately once per month).

Figure 3. This figure shows the joint probability of distress (JPD), for two copula models. Both models allow for a structural break on April 8, 2009. Distress is defined using the 99% quantile, and we use the probability of a large proportion of firms being in distress as a measure of systemic risk. We define the “joint probability of distress” as

\[ JPD_{t,k} \equiv \Pr \left[ \left( \frac{1}{N} \sum_{i=1}^{N} D_{i,t+250} \right) \geq \frac{k}{N} \right], \]

where \( k \) is a user-chosen threshold for what constitutes a “large” proportion of the \( N \) firms. We use \( k = 30 \), and the results corresponding to \( k = 20 \) and \( k = 40 \) are qualitatively similar. The JPD is only estimable via simulations from our model, and we obtain these using 10,000 simulations. Given the computational burden, we compute estimates only every 20 trading days (approximately once per month).

The estimated joint probability of distress is presented in Figure 3. For comparison, we also include the JPD implied by a “baseline” copula model, the constant Normal copula. (This copula allows for a break at the Big Bang, and so is constant only within each subperiod.) We observe that the JPD implied by the factor copula with GAS dynamics spikes up in late 2007, roughly doubling in value. It rises even further in early 2008, reaching a peak of around 1%. This implies that at the height of the financial crisis, there was around a 1% chance that at least 30 of these 100 firms would be simultaneously in distress, defined as a CDS spread lying in its upper 1% tail. The joint probability of distress declines in mid-2009, though remains somewhat higher than before the crisis. This is consistent with results for systemic sovereign default risk in the U.S. and Europe reported in Ang and Longstaff (2013) and Lucas, Schwaab, and Zhang (2014).

Comparing the estimates of JPD from the two copula models, we see that the constant normal copula estimate is almost uniformly lower than that from the factor copula in the precrisis period, and substantially lower from mid-2007 until the Big
Bang, when the copula parameter was allowed to change. In the second sub-sample the two estimates are closer, although the JPD implied by the normal copula is generally lower than from the factor copula. These two estimates of JPD reveal the large variation in estimates of systemic risk that can be obtained depending on the assumed shape of the copula, and the assumed form of the dynamics of the copula.

5.2 Expected Proportion in Distress

Our second measure of systemic risk more fully exploits the ability of our dynamic copula model to capture heterogenous dependence between individual CDS spread changes. For each firm \(i\), we compute the expected proportion of firms in distress conditional on firm \(i\) being in distress:

\[
\text{EPD}_{i,t} = E_t \left[ \frac{1}{N} \sum_{j=1}^{N} D_{j,t+250} \right] D_{i,t+250} = 1. \tag{21}
\]

The minimum value this can take is \(1/N\), as we include firm \(i\) in the sum, and the maximum is one. A version of this measure was proposed in Hartmann, Straetmans, and de Vries (2006). We use the same indicator for distress as in the previous section (Equation (19)). This measure of systemic risk is similar in spirit to the CoVaR measure proposed by Adrian and Brunnermeier (2016), in that it looks at distress “spillovers” from a single firm to the market as a whole. (Note that the summation sign in Equation (21) weights all firms equally; a simple extension would allow firms to be weighted by their market capitalization, debt-to-equity ratio, or any other measurable characteristic of a firm.)

In the upper panel of Figure 4, we summarize the results from the EPD estimates, and present the average, and 20% and 80% quantiles of this measure across the 100 firms in our sample. We observe that the average EPD is around 30% in the precrisis period, rising to almost 60% in late 2008, and returning to around 40% in the last year of our sample. Thus, this figure, like the JPD plot in Figure 3, is also suggestive of a large increase in systemic risk around the financial crisis, and higher level of systemic risk in the current period than in the precrisis period.

The expected proportion in distress measure enables us to identify firms that are more strongly correlated with market-wide distress than others. When the EPD is low for a given firm, it reveals that distress for that firm is not a signal of widespread distress, that is, firm \(i\) is more idiosyncratic. Conversely, when the EPD is high, it reveals that distress for this firm is a sign of widespread distress, and so this firm is a “bellwether” for systemic risk. To illustrate the information from individual firm EPD estimates, Table 6 presents the top five and bottom five firms according to their EPD on three dates in our sample period, the first day (January 2, 2006), a middle day (January 26, 2009), and the last day (April 17, 2012). We note that SLM Corporation (“Sallie Mae,” in the student loan business) appears in the “least systemic” group on all three dates, indicating that periods in which it is in distress are, according to our model, generally unrelated to periods of wider distress. Marsh and McLennan (which owns a collection of risk, insurance, and consulting firms) and Baxter International (a bioscience and medical firm) each appear in the “most systemic” group for two out of three dates.

Table 6 also provides information on the spread of EPD estimates across firms. At the start of our sample the least systemic firms had EPDs of 2 to 3, indicating that only one to two other firms are expected to be in distress when they are in distress. At the end of our sample, the least systemic firms had EPDs of 8 to 12, indicating a wider correlation of distress even among the least correlated. A similar finding is true for the most systemic firms: the EPDs for the most systemic firms rise from 48–53 at the start of the sample to 84–94 at the end. Thus, there is a general increase in the correlation between firm distress over this sample period.

A particular focus in the growing literature on systemic risk is on spillovers of distress from the financial sector to the nonfinancial, or “real,” sector, see Adrian and Brunnermeier (2016) and Acharya et al. (2016), for example. Our collection of 100 firms contains 16 financial firms and 84 nonfinancial firms, and we can examine the “expected proportion in distress” across these two classifications. The lower panel of Figure 4 presents estimates of the expected proportion of financial/nonfinancial firms in distress, conditional on one financial/nonfinancial firm
being in distress. This generates four time series of EPD, and allows us to look at spillovers from one sector to the other, and spillovers within a sector. This panel reveals substantial heterogeneity in the sensitivity of firms to other firms being in distress. In particular, we see that a substantially higher proportion of financial firms are expected to be in distress conditional on a nonfinancial firm being in distress, than the other way around. At the peak of the financial crisis, these proportions were 80% and 40%, respectively, and more recently have averaged around 60% and 30%. This might be taken as somewhat reassuring, as it indicates that while the probability of distress in the “real” sector does increase conditional on distress in the financial sector, the sensitivity is strongest in the other direction. We also note the relatively high sensitivity of financial firms to other financial firms being in distress, indicating some intraindustry contagion.

6. CONCLUSION

Motivated by the growing interest in measures of the risk of systemic events, this article proposes new flexible yet parsimonious models for time-varying high-dimensional distributions. We use copula theory to combine well-known models for univariate distributions with new models of the conditional dependence structure (copula) to obtain dynamic joint distributions. Our proposed new dynamic copula models can be applied in dimensions of 100 or more, which is much greater than commonly considered in the literature. These models draw on successful ideas from the literature on dynamic modeling of high-dimensional correlation matrices (e.g., Engle 2002) and on recent work on models for general time-varying distributions (Creal, Koopman, and Lucas 2013). We propose a “variance targeting” type estimator for this class of dynamic copulas to dramatically reduce the number of parameters to be estimated by numerical optimization.

We apply our model to a detailed analysis of a collection of 100 credit default swap (CDS) spreads on U.S. firms. The CDS market has expanded rapidly in recent years, and yields of 100 credit default swap (CDS) spreads on U.S. firms. The CDS market has expanded rapidly in recent years, and yields of 100 credit default swap (CDS) spreads on U.S. firms. The CDS market has expanded rapidly in recent years, and yields of 100 credit default swap (CDS) spreads on U.S. firms. The CDS market has expanded rapidly in recent years, and yields of 100 credit default swap (CDS) spreads on U.S. firms.

### APPENDIX: PROOF OF PROPOSITION 1

#### Proof of Proposition 1.

(i) The evolution equation for $\lambda_{it}$ in Equation (8) and stationarity of $\rho_{X}$, which holds by assumption 1(b), implies $\omega_t = E[\log \lambda_{it}] (1 - \beta)$, and we can rewrite our GAS Equation (8) in “variance targeting” form:

$$\log \lambda_{it} = E[\log \lambda_{it}] (1 - \beta) + \beta \log \lambda_{i,t-1} + \alpha s_{i,t-1}. $$

(A.1)

The objective of this proposition is to find an estimate of $E[\log \lambda_{it}]$ based on observable data, thus reducing the number of parameters to be estimated numerically. Next, note that linear correlations are given by

$$\rho_{ij}^L \equiv \text{Corr} [X_i, X_j] = \frac{\lambda_{i} \lambda_{j}}{\sqrt{1 + \lambda_{i}^2} \sqrt{1 + \lambda_{j}^2}} \equiv g (\lambda_{i}, \lambda_{j})$$

and

$$\text{R}_X^L \equiv \text{Corr} [X] = G (\lambda).$$

(A.2)

By assumption 1(a), this is an exactly-(N = 3) or over-(N > 3) identified system, as we have N parameters $\lambda \equiv \{\lambda_1, \ldots, \lambda_N\}$ and $N (N - 1)/2$ correlations. By Assumption 1(d), we have a corresponding exactly- or over-identified system for the Spearman rank correlation matrix:

$$\text{R}_X = \psi (\text{R}_X^L) = \psi (\text{G} (\lambda)).$$

(A.3)

(In a slight abuse of notation, we let $\psi (\text{R}_X^L)$ map the entire linear correlation matrix to the rank correlation matrix.) Define the exponential of the inverse of the function $\psi \circ G$ as $H$, so that

$$\log \lambda = H (\rho_{X}),$$

where $\rho_{X} \equiv \text{vech} (\text{R}_X^L)$. The function $H$ is not known in closed form but can be obtained by a simple and fast optimization problem:

$$H (\rho_{X}) = \arg \min (\text{vech} [\psi (G (\exp a))] - \rho_{X})^2 \times (\text{vech} [\psi (G (\exp a))] - \rho_{X}).$$

(A.4)

Note that under Assumption 1(a), there is no error in this optimization problem; this is just a means of recovering $H$ from $\psi \circ G$.

### Table 6. Estimates of systemic risk

<table>
<thead>
<tr>
<th>EPD</th>
<th>Firm</th>
<th>EPD</th>
<th>Firm</th>
<th>EPD</th>
<th>Firm</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 Jan</td>
<td>53 Marsh &amp; McLennan</td>
<td>78</td>
<td>Lockheed Martin</td>
<td>94</td>
<td>Wal-Mart</td>
</tr>
<tr>
<td>2</td>
<td>50 Hewlett-Packard</td>
<td>77</td>
<td>Campbell Soup</td>
<td>88</td>
<td>Baxter Int’l</td>
</tr>
<tr>
<td>3</td>
<td>50 IBM</td>
<td>75</td>
<td>Marsh &amp; McLennan</td>
<td>88</td>
<td>Walt Disney</td>
</tr>
<tr>
<td>4</td>
<td>49 Valero Energy</td>
<td>75</td>
<td>Baxter Int’l</td>
<td>87</td>
<td>Home Depot</td>
</tr>
<tr>
<td>5</td>
<td>48 Bristol-Myers Squibb</td>
<td>74</td>
<td>Goodrich</td>
<td>84</td>
<td>McDonald’s</td>
</tr>
</tbody>
</table>

NOTE: This table presents the five firms with the largest estimated expected proportion in distress (EPD), defined in Equation (21), and the five firms with the smallest EPD, for three dates in our sample period.
This is the GMM analog to the usual method-of-moments estimator used in variance targeting. Under Assumption 1(c), the function $H(\rho_X)$ is linear, so $E[\log \lambda_t] = E[H(\rho_X)] = H(\bar{\rho}_X)$, where $\lambda_t \equiv E(\rho_X)$. Finally, we exploit the fact that RankCorr $[X]$ is identical to RankCorr $[\eta]$ by Assumption 1(a) and Theorem 5.1.6 of Nelsen (2006). So we obtain $E[\log \lambda_t] = H(\bar{\rho}_X) = H(\hat{\rho}_X)$.

(ii) We use our "VT estimator" the sample analog of the above expression: $\hat{\log \lambda} = H(\hat{\rho}_X)$. Note that, since the marginal distributions of $\eta$ are known, sample rank correlations are linear functions of a sample moment, see Nelsen (2006, chap. 5), for example,

$$\hat{\rho}_{ij, \eta} = -3 + \frac{1}{T} \sum_{t=1}^{T} F_i(\eta_{t,i}) F_j(\eta_{t,j}).$$

Our estimate of $E[\log \lambda_t]$ is obtained in Equation (A.4) as

$$\log \lambda = \arg \min_a \tilde{m}_T(a) \tilde{m}_T(a),$$

where $\tilde{m}_T(a) \equiv \text{vech} \{ \psi(G(\exp(a))) - \hat{\rho}_X \}$. The element of $\tilde{m}_T$ corresponding to the $(i, j)$ element of the correlation matrix is

$$\tilde{m}_{i,j}(a) = [\psi(G(\exp(a)))]_{i,j} + 3 - \frac{1}{T} \sum_{t=1}^{T} F_i(\eta_{t,i}) F_j(\eta_{t,j}).$$

Thus, $\log \lambda$ is a standard GMM estimator for $N \geq 3$.

SUPPLEMENTARY MATERIALS

The supplemental appendix contains additional details on the implementation of the models in this article, as well as some additional simulation results.

ACKNOWLEDGMENTS

The authors thank the editor, associate editor, and two referees, as well as Christian Brownlees, Drew Creal, Rob Engle, Jeff Russell, Flavio Ziegelmann, and seminar participants at the 2016 AEA meeting, Bank of Japan, Chicago, Duke, International Workshop on Financial Econometrics in Natai, INFORMS 2015, Monash, Montréal, National Graduate Institute for Policy Studies (Tokyo), NYU, Sveriges Riksbank, Tinbergen Institute, Toulouse School of Economics, UNC-Chapel Hill, UNSW, and UTS for helpful comments. An appendix containing additional results and a MATLAB toolbox for this article is available at http://www.econ.duke.edu/~ap7172/research.html. The views in this article are those of the authors and do not necessarily represent the views or policies of the Board of Governors of the Federal Reserve System or its staff.

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