Not for Publication Online Supplemental Appendix to:

Granular Betas and Risk Premium Functions

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- Section S1 provides estimates of the expected returns on different industry portfolios that may be attributed to each of the factors in the granular CAPM, FF3, FF3+Mom and FF5 models.
- Section S2 presents the risk premium function estimates for the granular FF3 model obtained by splitting the full sample into the three vicennia: 1960-1970s, 1980-1990s, 2000-2010s.
- Section S3 presents additional results on multi-dimensional partitions of the FF3+Mom and FF5 models, mirroring the results for the FF3 model discussed in Section 5.2 of the main part of the paper.

S1. Expected returns by industry

Table S.1: **Expected returns by industry.** This table present the total expected returns per factor, by industry. The expected return is based on the functional risk premium estimates at $G = 64$ along with cross-sectional and time-series averaged beta functions.

	CAPM	FF3			FF3+Mom				FF5				
	MKT	MKT	SMB	HML	MKT	SMB	HML	MOM	MKT	SMB	HML	RMW	CMA
All	3.923	3.691	1.013	0.369	3.777	0.987	0.085	0.228	2.339	1.110	0.247	-0.006	0.055
NoDur	3.211	2.929	0.097	0.202	2.984	0.125	0.063	0.114	1.932	0.304	-0.008	0.048	0.174
Durbl	4.347	4.199	1.718	0.431	4.276	1.579	0.093	0.306	2.711	1.977	0.333	0.027	0.093
Manuf	4.211	4.103	1.089	0.435	4.210	1.023	0.084	0.309	2.633	1.278	0.273	0.005	0.126
Enrgy	4.021	4.086	-0.229	0.638	4.431	-0.394	0.045	0.140	2.709	-0.093	0.203	-0.019	0.331
HiTec	4.753	4.171	1.598	-0.395	4.113	1.619	-0.178	0.282	2.511	1.414	-0.465	-0.069	-0.233
Telcm	3.467	3.194	-0.225	0.477	3.251	-0.193	0.129	0.219	2.020	-0.268	0.315	0.032	0.169
Shops	3.859	3.588	1.766	0.414	3.669	1.737	0.132	0.368	2.340	2.201	0.319	0.022	0.106
Hlth	3.821	3.344	0.467	-0.221	3.506	0.456	-0.137	0.042	2.170	0.571	-0.601	0.003	0.094
Utils	2.297	2.451	-0.237	0.738	2.508	-0.213	0.249	0.123	1.538	-0.343	0.682	0.025	0.124
Finance	4.178	3.933	1.233	0.956	3.960	1.268	0.346	0.211	2.405	1.227	1.111	0.005	-0.099
Other	4.223	3.841	2.275	0.174	3.981	2.190	-0.011	0.256	2.445	2.488	0.020	-0.073	0.033

S2. Risk premium functions by vicennia

Figure S.1 presents the risk premium function estimates for the granular FF3 model obtained by splitting the full sample into the three vicennia: 1960-1970s, 1980-1990s, 2000-2010s. Perhaps most striking is the change in the level and shape of the risk premium function for the size factor. While the function is uniformly above zero and mostly concave for the earlier 1960-1970s time period, it is predominantly below zero and mostly convex for the latter forty years of the sample. These findings are, of course, also consistent with the extant literature, which reports that the size effect was positive and significant up until around 1980 (Banz, 1981; Reinganum, 1981), and then large disappeared afterwards (Schwert, 2003; Ahn et al., 2019). Similarly, the differential pricing of up- and down-side betas (Ang et al., 2006) also appear to have weakened considerable over the years, as evidence by the first panel in the figure (see also Levi and Welch, 2020).

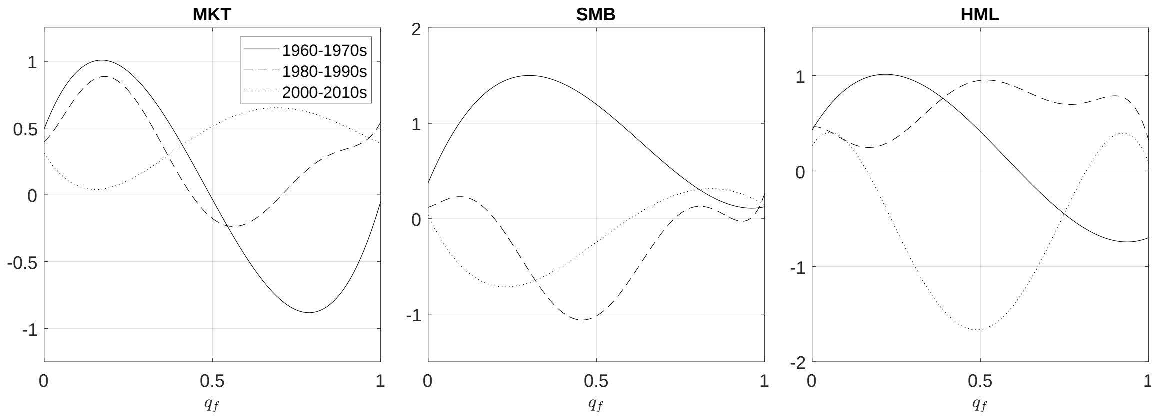

Figure S.1: **Risk premium function estimates by vicennia.** The figure presents the FF3 risk premium function estimates using functional betas with $G = 64$ for non-overlapping 20-year subsamples.

S3. Additional multi-dimensional partition results

This appendix presents additional results for multi-dimensional partitions of the FF3+Mom and FF5 models, mirroring the results for the FF3 model discussed in Section 5.2.

In particular, as the results in Table S.2 show, while it is possible to improve on the CAPM by allowing for one-dimensional partitions, additional improvements are obtained by considering four-dimensional partitions that draw on information from the size, value and momentum factors, with the R^2s increasing from 3.1% to 3.8% to 5.2%, respectively. The FF3+Mom four-factor model, however, does not perform better by allowing four-way partitions. Instead, the gains from considering granular betas for that model are best captured using simpler one-dimensional partitions.[1]

Further extending the results in the main part of the paper, Table S.3 report the results from additional five-dimensional partitions. Since it is not computationally feasible to consider the FF5 model with five-way partitions, we only consider such partitions for the CAPM.[2] The results in the table show that while one-way partitions lead to large improvements in out-of-sample fit, as discussed previously, using the granular betas based on five-dimensional partitions the optimal order of the Legendre polynomial is zero. That is, our validation sample finds that the optimal risk premium function is flat, so that the granular and non-granular models result in identical out-of-sample forecast performance. This therefore suggests that granular betas based on five-dimensional partitions are simply too "noisy" to be useful for out-of-sample forecasting.

[1]Note that the validation-sample optimal order of the Legendre polynomial in the most flexible four-way partition of the FF3+Mom model, reported in the bottom-right panel of Table S.2, equals one for all four factors. This implies that the risk premium function is linear in each of the different variables, and that only one-way partitioning information is employed. In other words, the validation sample tuning of the hyperparameters reduces the multi-dimensional partitions to one-dimensional partitions.

[2]Given the lack of any gains from multi-dimensional partitions for the three- and four-factor models evident in Table 3 and Table S.2, it would also be highly surprising to observe any gains for this even more refined model.

Table S.2: **One-way versus four-way partitions for granular betas.** This table presents a comparison of the out-of-sample R^2 and cross-validated optimal polynomial order for the CAPM and Carhart (1997) models, using either one-way (left panel) or four-way (right) panel partitions. The R^2 values in the left panel correspond to those in Table 1 and are reported here for ease of comparison. The order of the polynomials (G) compared in each row are selected so that the total number of granular betas (denoted #Gβ in the first column) is the same for the one-way and four-way partitions, aside from the bottom row in Panel A, where the number of one-way partitioned granular betas is the smaller of the two numbers given in the row label.

	One-way partitions			*Four-way partitions*		
#Gβ	G	Opt. order	R^2	G	Opt. order	R^2
Panel A: CAPM						
1	1	0	3.145	1	0	3.145
16	16	8	3.805	2	2,2,2,2	5.197
64 or 256	64	8	3.840	4	2,2,2,2	5.219
Panel B: FF3+Mom						
4	1	0	5.204	1	0	5.204
64	16	6,4,3,3	5.770	2	2,2,2,2	5.826
192	64	3,6,4,3	5.787	4	1,1,1,1	5.777

Table S.3: **One-way versus four-way partitions for granular betas.** This table presents a comparison of the out-of-sample R^2 and cross-validated optimal polynomial order for the CAPM, using either one-way (left panel) or five-way (right) panel partitions, where the five-way partitions are formed using the factors in the Fama and French (2015) five-factor model. The R^2 values in the left panel correspond to those in Table 1 and are reported here for ease of comparison. The order of the polynomials (G) compared in each row are selected so that the total number of granular betas (denoted #Gβ in the first column) is the same for the one-way and five-way partitions, aside from the bottom row, where the number of one-way partitioned granular betas is the smaller of two numbers given in the row label.

	One-way partitions			*Five-way partitions*		
#Gβ	G	Opt. order	R^2	G	Opt. order	R^2
1	1	0	3.145	1	0	3.145
32	32	8	3.795	2	0,0,0,0,0	3.145
64 or 1024	64	8	3.840	4	0,0,0,0,0	3.145