Supplementary Appendix to "A delegation-based theory of expertise"

Attila Ambrus, Volodymyr Baranovskyi and Aaron Kolb

May 20, 2017

This supplement provides welfare results not contained in the main text and a proof of Lemma A.1. For small bonuses, a mixed equilibrium exists if and only if a max equilibrium exists; if so, it is unique. For large bonuses, we find a unique candidate for mixed equilibrium and show that mixed and min equilibria cannot co-exist. Also, we give an example for equal biases, where this candidate is indeed a mixed equilibrium. However, when biases are different enough and the bonus is high, a mixed equilibrium does not exist. Though a general analytical comparison is infeasible, we show that mixed equilibria are inferior to min equilibrium or simple delegation in various special cases.

S.1 Equilibria with the Principal Mixing

Here we provide a partial characterization of equilibria in which the experts play constant markup strategies and the principal mixes between choosing the lower versus the higher offer. Recall that such equilibria belong to $\{(k_1, k_2, a = p\min(a_1, a_2) \oplus (1-p)\max(a_1, a_2)) : k_1 + k_2 = 0, p \in (0, 1)\}$.

Proposition S.1. *If experts follow constant markup strategies with markups k_1 and k_2, and the principal chooses lower offer with probability p, then the expected payoff of expert i is:*

$$U_i(k_i, k_j, p) = pU_i(k_i, k_j, L) + (1-p)U_i(k_i, k_j, H)$$

We divide the mixed equilibrium case into two subcases: $B \leq 2\sigma^2$ and $B \geq 2\sigma^2$. As it will be seen later, the properties of the mixed equilibrium on these intervals are different.

S.1.1 Mixed equilibrium for low bonuses

When $B \leq 2\sigma^2$, loss-minimizing incentives outweigh bonus-earning incentives. The next lemma states that if the principal commits to a mixed strategy, the game between experts has a unique equilibrium in constant markup strategies. The proof requires that $B \leq 2\sigma^2$ and therefore cannot be automatically extended to a general case. As p increases, expert strategies become closer to those in min equilibrium. Interestingly, $k_1(p)$ and $k_2(p)$ individually need not be monotonically increasing. For $p \geq \frac{1}{2}$ (resp. $p \leq \frac{1}{2}$), $k_1(p)$ (resp. $k_2(p)$) is increasing, but for complementary p, the behavior depends on other parameter values.

Lemma S.1. *Consider the game played between experts given a fixed principal strategy p. For $B \leq 2\sigma^2$, this game has a unique equilibrium in constant markup strategies, characterized by differentiable markup functions $k_1(p)$ and $k_2(p)$, with $k_1(p) + k_2(p)$ increasing.*

Proof. In what follows, define

$$
\begin{aligned}
v(z) &:= \frac{f(z)}{1 - F(z)} \\
w(z) &:= \frac{f(z)}{F(z)} \\
J_v(z;p) &:= \frac{1 - F(z)}{\frac{p}{2p-1} - F(z)} \\
J_w(z;p) &:= \frac{F(z)}{\frac{1-p}{2p-1} + F(z)} \\
H(z;p) &:= \frac{f(z)}{\frac{p}{2p-1} - F(z)} = v(z)J_v(z) \\
K(z;p) &:= \frac{f(z)}{\frac{1-p}{2p-1} + F(z)} = w(z)J_w(z)
\end{aligned}
$$

where f and F are the PDF and CDF of $N(0, 2\sigma^2)$.

Note that for all $z \in \mathbb{R}$ and $p \in [0,1]$, we have the identities $H(z; 1-p) \equiv -K(z;p)$ and $H(-z;p) \equiv K(z;p)$.

Given p, the experts' FOCs are

$$
\begin{aligned}
k_1 &= b_1 + \left(\sigma^2 - \frac{B}{2}\right) \frac{f(z)(2p-1)}{p(1-F(z)) + (1-p)F(z)} && \text{(S.1)} \\
&= b_1 + \left(\sigma^2 - \frac{B}{2}\right) H(z;p) && \text{(S.2)} \\
k_2 &= b_2 + \left(\sigma^2 - \frac{B}{2}\right) \frac{f(z)(2p-1)}{pF(z) + (1-p)(1-F(z))} && \text{(S.3)} \\
&= b_2 + \left(\sigma^2 - \frac{B}{2}\right) K(z;p), && \text{(S.4)}
\end{aligned}
$$

where $z = k_1 - k_2$. Note that for $p = \frac{1}{2}$, we have simply $k_1 = b_1$ and $k_2 = b_2$, and $H(z;p)$ and $K(z;p)$ are continuous in p for all $p \in [0,1]$. Henceforth, we consider $p \neq \frac{1}{2}$.

Combining (S.1) and (S.3) yields

$$
D(z;p) := b_1 - b_2 + \left(\sigma^2 - \frac{B}{2}\right) [H(z;p) - K(z;p)] - z = 0. \tag{S.5}
$$

It is easy to verify that the $D(z;p)$ is symmetric in p about $p = \frac{1}{2}$, so it suffices for the moment to consider only $p \in (\frac{1}{2}, 1)$. That a solution to (S.5) exists is evident upon examining the limiting behavior; as z approaches $+\infty$ or $-\infty$, $D(z;p)$ approaches $+\infty$ or $-\infty$, respectively. Every solution to (S.5) determines a unique pair (k_1, k_2) that solves (S.1) and (S.3). We now show that there is a unique solution z^* to (S.5); it suffices to show that the $D(z;p)$ has derivative strictly less than

0. We have $\frac{\partial H(z;p)}{\partial z} := v'(z)J_v(z) + v(z)J_v'(z)$. Now $v'(z) > 0$ by Lemma 1 and since $p \geq \frac{1}{2}$, $J_v(z) < 1$ and $J_v'(z) < 0$. Thus by Lemma 1, $\frac{\partial H(z;p)}{\partial z} < v'(z) < \frac{1}{2\sigma^2}$. By similar reasoning, $\frac{\partial K(z;p)}{\partial z} > w'(z) > -\frac{1}{2\sigma^2}$. If $\frac{\partial [H(z;p)-K(z;p)]}{\partial z} < 0$, the claim immediately follows, and if not, we have $\left(\sigma^2 - \frac{B}{2}\right)\frac{\partial [H(z;p)-K(z;p)]}{\partial z} < \sigma^2 \frac{2}{2\sigma^2} = 1$, as desired.

It remains to show that for the unique solution (k_1^*, k_2^*) to (S.1) and (S.3), each expert is indeed best-responding to the other. Consider two cases.

Case I: $p > \frac{1}{2}$. By the arguments above, the right side of (S.1) has derivative w.r.t. z (and hence k_1) less than 1. Since the labeling of experts is arbitrary, the second order condition is satisfied for each expert.

Case II: $p < \frac{1}{2}$. By the identity $H(z;1-p) = -K(z;p)$, (S.3) becomes $k_2 = b_2 - \left(\sigma^2 - \frac{B}{2}\right)H(z;1-p)$. By familiar arguments, the right side has derivative w.r.t. k_2 less than 1, and since experts are labeled arbitrarily, second order conditions hold.

Clearly, (S.1) and (S.3) are equations of differentiable functions of k_1, k_2, and p, and are non-constant in the k_i. By the implicit function theorem, there exist differentiable functions $k_1(p), k_2(p)$ that solve these equations for each p. Finally, we show that $k_1(p) + k_2(p)$ is increasing. It is useful now to label players so that $b_1 > b_2$, and thus $z^*(p) > 0$. Combining (S.1) and (S.3), we obtain

$$S(k_1, k_2, z; p) := b_1 + b_2 + \left(\sigma^2 - \frac{B}{2}\right)[H(z;p) + K(z;p)] - (k_1 + k_2) = 0 \tag{S.6}$$

Note that $z^*(p)$ is decreasing in p for $p < \frac{1}{2}$ and increasing for $p > \frac{1}{2}$. To see this, consider (S.5). Since $D(z;p)$ is symmetric in p about $p = \frac{1}{2}$, it suffices to consider $p > \frac{1}{2}$. We have

$$\frac{\partial [H(z;p) - K(z;p)]}{\partial p} = \frac{f(z)}{(2p-1)^2}\left[\left(\frac{p}{2p-1} - F(z)\right)^{-2} - \left(\frac{1-p}{2p-1} + F(z)\right)^{-2}\right].$$

Now for all $z > 0$, $F(z) > \frac{1}{2}$ and thus $\frac{1-p}{2p-1} + F(z) > \frac{p}{2p-1} - F(z) > 0$. It follows that $\frac{\partial D(z;p)}{\partial p} > 0$. By earlier claims, $D(z;p)$ is decreasing in z, so $z^*(p)$ must be increasing in p for $p > \frac{1}{2}$. By symmetry, $z^*(p)$ is decreasing in p for $p < \frac{1}{2}$. Let $C = \sigma^2 - B/2 \geq 0$. By totally differentiating (S.5) w.r.t p, we obtain

$$z'(p) = \frac{C(H_p - K_p)}{1 - C(H_z - K_z)}. \tag{S.7}$$

From (S.6), we have

$$\begin{aligned} k_1'(p) + k_2'(p) &= C[(H_z + K_z)z'(p) + H_p + K_p] \\ &= C(H_z + K_z)\frac{C(H_p - K_p)}{1 - C(H_z - K_z)} + C(H_p + K_p). \end{aligned}$$

The above is nonnegative if and only if

$$\begin{aligned} &(H_z + K_z)C(H_p - K_p) + (H_p + K_p)(1 - C(H_z - K_z)) \geq 0 \\ \iff &H_p + K_p + 2CK_zH_p - 2CH_zK_p \geq 0 \end{aligned} \tag{S.8}$$

The LHS of (S.8) is at least $H_p + K_p + 2Cw'(z)H_p - 2Cv'(z)K_p$, which by Lemma 1 is positive, as desired.

For $p < \frac{1}{2}$, we use the substitution $p' = 1 - p$ and apply the identity $H(z;p) \equiv K(z;1-p)$; we obtain $k_1(p) + k_2(p) - b_1 - b_2 = -(k_1(p') + k_2(p') - b_1 - b_2)$. Differentiating both sides and noting that $\frac{dp'}{dp} = -1$, we get that $k_1'(p) + k_2'(p) = k_1'(1-p) + k_2'(1-p) > 0$. □

The result above allows us to determine the existence of mixed equilibrium; the principal's indifference condition is simply $k_1(p) + k_2(p) = 0$.

Proposition S.2. *For $B \leq 2\sigma^2$, a mixed strategy equilibrium without commitment exists if and only if a max equilibrium exists. When it exists, it is unique, and $z^* \geq b_1 - b_2 \geq 0$, with equality if and only if $b_1 = b_2$.*

Proof. Consider an equilibrium candidate defined by p, $k_1(p)$, and $k_2(p)$. If $p \in (0,1)$, principal optimality is equivalent to $k_1(p) + k_2(p) = 0$. Note that an min equilibrium is guaranteed to exist for these parameters; that is, $k_1(1) + k_2(1) > 0$. By Lemma S.1, $k_1(p) + k_2(p)$ is continuous and increasing in p, so there exists a unique $p^* \in (0,1)$ such that $k_1(p^*) + k_2(p^*) = 0$ if and only if $k_1(0) + k_2(0) < 0$. □

We can conclude that the conditions for the mixed equilibrium existence on the interval $B \in [0, 2\sigma^2]$ coincide with the ones for the existence of a max equilibrium, that were already explored in the main text.

S.1.2 Mixed equilibrium for high bonuses

We now consider high bonuses, $B > 2\sigma^2$. In the next Proposition we find the critical point, that is a unique candidate for a mixed equilibrium, given any set of parameters.

Proposition S.3. *Let $B \geq 2\sigma^2$ and let z be the unique solution to the equation*

$$(\frac{z}{2} - b_1)(-\frac{z}{2} - b_2)(1 - 2F(z)) = (\frac{B}{2} - \sigma^2)(z - b_1 + b_2)f(z) \tag{S.9}$$

on the interval $(-2b_2, 2b_1)$.
Then $(\frac{z}{2}, -\frac{z}{2}, a = p\min(a_1, a_2) \oplus (1-p)\max(a_1, a_2))$ is the only candidate for a mixed equilibrium. It is an equilibrium only if $0 \leq z \leq b_1 - b_2$ and

$$\left(\frac{B}{2} - \sigma^2\right)\left[\frac{f(z)}{1 - F(z)} + \frac{f(z)}{F(z)}\right] \geq b_1 + b_2. \tag{S.10}$$

Proof. If the principal chooses the lower offer with probability p, FOCs can be obtained by differentiating utilities from Proposition S.1:

$$0 = p\left[(k_1 - b_1)(1 - F(z)) + \left(\frac{B}{2} - \sigma^2\right)f(z)\right] + (1-p)\left[(k_1 - b_1)F(z) - \left(\frac{B}{2} - \sigma^2\right)f(z)\right]$$
$$0 = p\left[(k_2 - b_2)F(z) + \left(\frac{B}{2} - \sigma^2\right)f(z)\right] + (1-p)\left[(k_2 - b_2)(1 - F(z)) - \left(\frac{B}{2} - \sigma^2\right)f(z)\right]$$

In what follows, define

$$A := (k_1 - b_1)(1 - F(z)) + \left(\frac{B}{2} - \sigma^2\right) f(z),$$
$$E := (k_1 - b_1)F(z) - \left(\frac{B}{2} - \sigma^2\right) f(z),$$
$$C := (k_2 - b_2)F(z) + \left(\frac{B}{2} - \sigma^2\right) f(z),$$
$$D := (k_2 - b_2)(1 - F(z)) - \left(\frac{B}{2} - \sigma^2\right) f(z).$$

Combining FOCs with the principal's optimality condition, we get the following system of equations:

$$\begin{cases} k_1 + k_2 = 0 & \text{(S.11)} \\ p \in (0, 1) & \text{(S.12)} \\ pA + (1-p)E = 0 & \text{(S.13)} \\ pC + (1-p)D = 0 & \text{(S.14)} \end{cases}$$

Now notice that $A+E = k_1 - b_1$, $C + D = k_2 - b_2$. Then $A+E+C+D = k_1 + k_2 - (b_1 + b_2) = -(b_1 + b_2) \leq 0$. Therefore, either $A + E = k_1 - b_1 \leq 0$ or $C + D = k_2 - b_2 \leq 0$.

(S.13) and (S.14) are equivalent to:

$$pA + (1-p)E = (k_1 - b_1)[p(1 - F(z)) + (1-p)F(z)] + (2p-1)\left(\frac{B}{2} - \sigma^2\right) f(z) = 0$$
$$pC + (1-p)D = (k_2 - b_2)[pF(z)) + (1-p)(1 - F(z))] + (2p-1)\left(\frac{B}{2} - \sigma^2\right) f(z) = 0$$

Both expressions in square brackets are positive, therefore $sign(k_1 - b_1) = sign(k_2 - b_2)$. Hence $A + E = k_1 - b_1 \leq 0$, $C + D = k_2 - b_2 \leq 0$ and $(2p-1)\left(\frac{B}{2} - \sigma^2\right) \geq 0 \implies p \geq \frac{1}{2}$

Moreover, looking at the initial system of FOCs, we find that
$sign(A) = sign(C) = sign(B - 2\sigma^2), sign(E) = sign(D) = -sign(B - 2\sigma^2)$.

As $B > 2\sigma^2$, we get: $A > 0, E < 0, C > 0, D < 0$.

Also notice that in order to be an equilibrium the solution of the system (S.11)-(S.14) must also satisfy SOCs for both experts:

$$0 \geq p\left[-2(1 - F(z)) + 2(k_1 - b_1)f(z) + \frac{B}{2\sigma^2}(z)f(z)\right]$$
$$+ (1-p)\left[-2F(z) - 2(k_1 - b_1)f(z) - \frac{B}{2\sigma^2}(z)f(z)\right],$$
$$0 \geq p\left[-2F(z) + 2(k_2 - b_2)f(z) - \frac{B}{2\sigma^2}(z)f(z)\right]$$
$$+ (1-p)\left[-2(1 - F(z)) - 2(k_2 - b_2)f(z) + \frac{B}{2\sigma^2}(z)f(z)\right].$$

From (S.13) and (S.14) we have that:

$$\frac{p}{1-p} = -\frac{E}{A} = -\frac{D}{C}$$

Using this equality, we obtain:

$$A\left[-2(1-F(z)) + 2(k_1-b_1)f(z) + \frac{B}{2\sigma^2}zf(z)\right] - E\left[-2F(z) - 2(k_1-b_1)f(z) - \frac{B}{2\sigma^2}zf(z)\right] \le 0;$$
$$C\left[-2F(z) + 2(k_2-b_2)f(z) - \frac{B}{2\sigma^2}zf(z)\right] - D\left[-2(1-F(z)) - 2(k_2-b_2)f(z) + \frac{B}{2\sigma^2}zf(z)\right] \le 0$$

After transformations we receive the following system of inequalities, which is equivalent to SOCs:

$$2(k_1-b_1)^2 + \left(\frac{B}{2\sigma^2}-1\right)(k_1-b_1)z + B - 2\sigma^2 \ge 0 \tag{S.15}$$
$$2(k_2-b_2)^2 - \left(\frac{B}{2\sigma^2}-1\right)(k_2-b_2)z + B - 2\sigma^2 \ge 0 \tag{S.16}$$

It follows that the initial system of equations (S.11)-(S.14) is equivalent to:

$$\begin{cases} k_1 + k_2 = 0 & \text{(S.17)} \\ AD = EC & \text{(S.18)} \\ k_1 - b_1 \le 0 & \text{(S.19)} \\ k_2 - b_2 \le 0 & \text{(S.20)} \\ \dfrac{A}{1-F(z)} + \dfrac{C}{F(z)} > 0 & \text{(S.21)} \end{cases}$$

All the equations (S.17)-(S.21) were derived from the initial system (S.11)-(S.14). (S.19) and (S.20) guarantee negative signs of E and D correspondingly, together with (S.18) implying $sign(A) = sign(C)$. Then (S.21) guarantees us $A > 0, C > 0$. Therefore systems of equations (S.11)-(S.14) and (S.17)-(S.21) are equivalent. From (S.17), we get $k_1 = \frac{z}{2}$, $k_2 = -\frac{z}{2}$. Next, we put them in (S.18) and obtain (S.9):

$$\left(\frac{z}{2}-b_1\right)\left(-\frac{z}{2}-b_2\right)(1-2F(z)) = \left(\frac{B}{2}-\sigma^2\right)(z-b_1+b_2)f(z).$$

Now consider three subcases:

a) $b_1 = b_2 = b \ge 0$.

Here (S.9) can be rewritten as:

$$\left(\frac{z}{2}-b\right)\left(-\frac{z}{2}-b\right)(1-2F(z)) = \left(\frac{B}{2}-\sigma^2\right)zf(z) \tag{S.22}$$

One solution of (S.22) is $z = 0$, which implies $k_1 = k_2 = 0$ and satisfies (S.10).

If we look for other solutions we get

$$g(z) = \frac{z}{4} - \frac{b^2}{z} + \left(\frac{B}{2} - \sigma^2\right)\frac{f(z)}{1-2F(z)} = 0.$$

$g(z)$ is increasing at $z \neq 0$, $g(-\infty) = -\infty$, $g(0-) = \infty$, $g(0+) = -\infty$, $g(\infty) = \infty$. Therefore, $g(z)$ has 2 roots: $z_1 < 0$ and $z_2 > 0$. However $g(-2b) > 0$, hence $z_1 < -2b$, that contradicts (S.20). By analogy $g(2b) < 0$, hence $z_2 > 2b$, that contradicts (S.19). Therefore, $z = 0$ is the only candidate for a mixed equilibrium.

b) $b_1 + b_2 = 0$. In this case FOCs imply $k_1 = b_1$, $k_2 = b_2$ and $p = \frac{1}{2}$, that satisfies (S.10).

c) $b_1^2 \neq b_2^2$. Here 0 and $b_1 - b_2$ are not roots of (S.9) and the last one can be rewritten as:

$$l(z) = \frac{z - b_1 + b_2}{4} - \frac{(b_1+b_2)^2}{4(z - b_1 + b_2)} + \left(\frac{B}{2} - \sigma^2\right)\frac{f(z)}{1-2F(z)} = 0. \tag{S.23}$$

$l(z)$ is increasing at $l \neq 0$ and $l \neq b_1 - b_2$; $l(-\infty) = -\infty$, $l(0-) = \infty$, $l(0+) = -\infty$, $l(b_1 - b_2-) = \infty$, $l(b_1 - b_2+) = -\infty$, $l(\infty) = \infty$. Therefore $l(z)$ has 3 roots: $z_1 < 0$, $0 < z_2 < b_1 - b_2$ and $z_3 > b_1 - b_2$. However, $l(2b_1) < 0$ and $z_3 > 2b_1$, which contradicts (S.19). Now consider 2 cases.

1) $b_2 \geq 0$. Here $l(-2b_2) > 0$ and $z_1 < -2b_2$, which contradicts (S.20).

2) $b_2 < 0$. Here $z_1 < 0 < -2b_2$, which contradicts (S.20).

Hence, $z = z_2$ is the only candidate for a mixed equilibrium. It is easy to see that z_2 satisfies (S.19) and (S.20). After we simplify (S.21), we get an equivalent (S.10). □

The following corollary collects observations from Theorems 2 and 3 and Propositions S.2 and S.3, requiring no proof.

Corollary S.1. *In any mixed, min, or max equilibrium, the player with higher bias applies a higher markup.*

In the following Theorem we show that for $B \geq 2\sigma^2$ the mixed equilibrium may exist only for $B \geq B_u$, implying that it can't coexist with the other types of equilibria.

Proposition S.4. *For $B \geq 2\sigma^2$, a mixed equilibrium may exist only if $B \geq B_m = B_u$.*

Proof. If $z^*(B)$ is the solution of (2), then B_u satisfies

$$\left(\frac{B_u}{2} - \sigma^2\right)\left[\frac{f(z^*(B_u))}{1 - F(z^*(B_u))} + \frac{f(z^*(B_u))}{F(z^*(B_u))}\right] = b_1 + b_2$$

B_u and the solution of (S.9) $z(B_u)$ satisfy (S.10) if and only if $z(B_u) \leq z^*(B_u)$. From (S.9) we get that $z(B_u) = z^*(B_u)$. Note that $z(B)$ is increasing, implying that the LHS of (S.10) is increasing in B. As a consequence, (S.10) is satisfied only if $B \geq B_u$ □

While Proposition S.4 gives a necessary condition for the existence of mixed equilibrium when $B > 2\sigma^2$, sufficient conditions are difficult to find; unlike $U_i(k_i, k_j, L)$ and $U_i(k_i, k_j, H)$, $U_i(k_i, k_j, p)$ is not a single-peaked function. The next example shows that even for $B \geq B_u$, mixed equilibrium does not always exist.

Note S.1. *$U_1(k_1, k_2, p)$ has 1 or 2 local maxima, both numbers are feasible depending on the parameters of the model.*

Proof. The marginal utility of expert 1 is equal to

$$U_1' = -2\left[(k_1 - b_1)(p(1 - F(k_1 - k_2)) + (1 - p)F(k_1 - k_2)) + (2p - 1)\left(\frac{B}{2} - \sigma^2\right) f(k_1 - k_2)\right] =$$
$$= -2\left[p(1 - F(k_1 - k_2)) + (1 - p)F(k_1 - k_2)\right]\left[k_1 - b_1 + (2p - 1)\left(\frac{B}{2} - \sigma^2\right) s(k_1 - k_2)\right],$$

where $s(x) = \frac{f(x)}{p(1-F(x))+(1-p)F(x)} > 0$.
Hence, $sign(U_1'(k_1)) = -sign\left[k_1 - b_1 + (2p - 1)\left(\frac{B}{2} - \sigma^2\right) s(k_1 - k_2)\right] = -sign(g(k_1))$
Then the number of critical points of U_1' is equal to number of zeros of $g(k_1)$.
Also notice that $g'(k_1) = 1 + (2p - 1)\left(\frac{B}{2} - \sigma^2\right) s'(k_1 - k_2)$
Differentiating, we get that $s'(t) = s(t)[(2p - 1)s(t) - \frac{t}{2\sigma^2}]$

1) For every $t \le 0 : s'(t) > 0$.

2) $s(t)$ tends to 0 as t tends to infinity, therefore exists $t_1 : s'(t_1) = 0$. But then for $t > t_1$: $(2p - 1)s(t) - \frac{t}{2\sigma^2}$ is decreasing and negative, implying $s'(t) < 0$.

3) $s''(t) = s(t)([(2p - 1)s(t) - \frac{t}{2\sigma^2}][2(2p - 1)s(t) - \frac{t}{2\sigma^2}] - 1)$
$s''(t_1) < 0$, $2(2p - 1)s(t) - \frac{t}{2\sigma^2}$ is strictly decreasing for $t > t_1$, then exists $t_2 > t_1 : 2(2p - 1)s(t_2) - \frac{t_2}{2\sigma^2} = 0$. $s''(t_2) < 0$ and for $t > t_2 : [(2p - 1)s(t) - \frac{t}{2\sigma^2}][2(2p - 1)s(t) - \frac{t}{2\sigma^2}] - 1$ is increasing and tends to infinity, therefore exists $t_3 : s''(t_3) = 0$ and for $t > t_3 : s''(t) > 0$.

Summing up, $s'(t)$ is positive for all $t < t_1$ and is negative for all $t > t_1$; is decreasing on (t_1, t_3) and increasing for $t > t_3$.
Returning to g, we see that this function potentially may have either 1 or 3 zeros. Therefore, U_1' can have either 1 local (and global) maximum or 2 local maximums and 1 local minimum. □

Proposition S.5. *If $b_1^2 > b_2^2 + 4\sigma^2$ and B is large enough, then there is no mixed equilibrium.*

Proof. It is enough to show that SOC for expert 1 is not satisfied. The SOC for expert 1 (S.15) is equivalent to

$$g(z) = z^2 - \left(\frac{4\sigma^2}{B} + 2\right) b_1 z + \frac{8b_1^2\sigma^2}{B} + \frac{4\sigma^2}{B}(B - 2\sigma^2) \ge 0$$

If B tends to infinity, the solution of the equation (S.23) z tends to $b_1 - b_2$. Calculating g at point $b_1 - b_2$, we get:

$$Bg(b_1 - b_2) = (B - 2\sigma^2)(b_2^2 - b_1^2 + 4\sigma^2) + 2\sigma^2(b_1 + b_2)^2$$

If $b_1^2 > b_2^2 + 4\sigma^2$ and B is large enough, then $g(b_1 - b_2)$ becomes negative. As a result, for $b_1^2 > b_2^2 + 4\sigma^2$ and large enough B SOC for expert 1 is not satisfied and $k_1^* = \frac{z}{2}$ is a point of local min for $U_1(k_1, k_2^*, p^*)$. □

The latter proposition shows that for some set of parameters $U_1(k_1, k_2^*, p^*)$ has one local minimum and two local maxima (following from its behavior at positive and negative infinity). In such situations it is not enough to check SOCs, since those can only confirm that (k_1^*, k_2^*, p^*) is a point of local, but not necessarily global maximum. To check for the latter, the two local maxima should be compared.

Next we consider the equal biases case: $b_1 = b_2 = b$.

Proposition S.6. *Consider* $b_1 = b_2 = b$.
If $B < 2\sigma^2 + 2\sqrt{\pi}\sigma b$*, then there is no mixed equilibrium.*
If $B \geq 2\sigma^2 + 2\sqrt{\pi}\sigma b$*, then* $(0, 0, a = p_M \min(a_1, a_2) \oplus (1 - p_M)\max(a_1, a_2))$*,*
where $p_M = \frac{1}{2} + \frac{b\sqrt{\pi}\sigma}{B-2\sigma^2} \geq \frac{1}{2}$*, is the only candidate for mixed equilibrium.*
If this equilibrium exists, each expert receives $U(0, 0, p_M) = \frac{B}{2} - \sigma^2 - b^2 - \frac{4b^2\sigma^2}{B-2\sigma^2}$ *and the principal gets* $V(0, 0, p_M) = -\sigma^2$.

Proof. If $b_1 = b_2 = b$, from the proof of Proposition S.3 we know that $z = 0$ is the only solution of (S.22) and $k_1^* = k_2^* = 0$ is the only candidate for mixed equilibrium. Returning to FOCs, we get:

$$2p_M - 1 = \frac{b}{(B - 2\sigma^2)f(0)} \in [0, 1]$$

which leads to the result formulated in Proposition. Next, from Propositions S.1, 2 and 3 we get both experts' utilities.

If the principal chooses minimal offer, then in state θ the principal's action a is distributed as $\theta + \eta$, where $\eta \sim \min(\epsilon_1, \epsilon_2)$; $\epsilon_1, \epsilon_2 \sim N(0, \sigma^2)$, ϵ_1 and ϵ_2 are independent. Integration by parts gives $E\eta = \frac{\sigma}{\sqrt{\pi}}$, $E\eta^2 = \sigma^2$. Therefore the principal's utility $V(0, 0, L) = -E(a - \theta)^2 = -E(\theta + \eta - \theta)^2 = -E\eta^2 = -\sigma^2$. By analogy, $V(0, 0, H) = -\sigma^2$, hence $V(0, 0, p_M) = -\sigma^2$ □

All the conditions mentioned before were necessary for the existence of a mixed equilibrium. Also we showed that if B is high enough and $b_1^2 - b_2^2 > 4\sigma^2$, there is no mixed equilibrium. Next, we proceed with an example of a mixed equilibrium existence.

Proposition S.7. *Consider* $b_1 = b_2 = b$*, when* $b \leq 1.05\sigma$.
If $B \geq 2\sigma^2 + 2\sqrt{\pi}\sigma b$*, then* $(0, 0, a = p_M \min(a_1, a_2) \oplus (1 - p_M)\max(a_1, a_2))$*,*
where $p_M = \frac{1}{2} + \frac{b\sqrt{\pi}\sigma}{B-2\sigma^2} \geq \frac{1}{2}$*, is the only mixed equilibrium.*

Proof. Consider Player 1's BR, given $k_2^* = 0$ and $p^* = \frac{1}{2} + \frac{b\sqrt{\pi}\sigma}{B-2\sigma^2}$:
$U_1'(k_1, k_2^* = 0, p^*) = -2[(k_1 - b)(p^*(1 - F(k_1 - k_2^*)) + (1 - p^*)F(k_1 - k_2^*)) + (2p^* - 1)\left(\frac{B}{2} - \sigma^2\right) f(k_1 - k_2^*)] = -2[(k_1 - b)(p^*(1 - F(k_1)) + (1 - p^*)F(k_1)) + b\sqrt{\pi}\sigma f(k_1)] = -2(p^*(1 - F(k_1)) + (1 - p^*)F(k_1))\left[k_1 - b + \frac{b\sqrt{\pi}\sigma f(k_1)}{p^*(1-F(k_1))+(1-p^*)F(k_1)}\right] = -2(p^*(1 - F(k_1)) + (1 - p^*)F(k_1))\left[k_1 - b + b\frac{s(k_1)}{s(0)}\right]$,
where $s(x) = \frac{f(x)}{p^*(1-F(x))+(1-p^*)F(x)}$

Therefore $U_1'(k_1) = 0$ if and only if $k_1 - b + b\frac{s(k_1)}{s(0)} = 0$ or, equivalently,

$$\frac{s(k_1)}{s(0)} + \frac{k_1}{b} = 1 \tag{S.24}$$

1) $k_1 < 0$. As $p^* \geq \frac{1}{2}$, $p^*(1 - F(k_1)) + (1 - p^*)F(k_1)$ is decreasing in k_1; $f(k_1) < f(0)$, hence $s(k_1) < s(0)$. Therefore, (S.24) has no negative solutions.

2) $k_1 > b$. Here $\frac{s(k_1)}{s(0)} + \frac{k_1}{b} > 0 + 1 = 1$, hence (S.24) has no solutions in this region.

3) $k_1 \in [0, b]$. Denote LHS of (S.24) by $g(k_1)$: $g(k_1) = \frac{s(k_1)}{s(0)} + \frac{k_1}{b}$.

Then $g'(k_1) = \frac{s'(k_1)}{s(0)} + \frac{1}{b}$. If parameters of the model are such that $g'(k_1) > 0$, or equivalently LHS of (S.24) is increasing on interval $k \in [0, b]$, then we conclude that (S.24) has the only solution $k_1^* = 0$.

$$\begin{aligned} g'(k_1) &= \frac{s'(k_1)}{s(0)} + \frac{1}{b} = \frac{s(k_1)[(2p^* - 1)s(k_1) - \frac{k_1}{2\sigma^2}]}{s(0)} + \frac{1}{b} \\ &\geq \frac{1}{b} - \frac{k_1 s(k_1)}{2\sigma^2 s(0)} = \frac{1}{b} - \frac{1}{2\sigma^2 s(0)} \frac{k_1 f(k_1)}{p^*(1 - F(k_1)) + (1 - p^*)F(k_1)}. \end{aligned}$$

Notice, that $k_1 f(k_1) \leq \sqrt{2}\sigma f(\sqrt{2}\sigma)$, $p^*(1 - F(k_1)) + (1 - p^*)F(k_1) = (2p^* - 1)(1 - F(k_1)) + 1 - p \geq (2p^* - 1)(b)) + 1 - p$, therefore:

$$g'(k_1) \geq \frac{1}{b} - \frac{1}{2\sigma^2 s(0)} \frac{k_1 f(k_1)}{p^*(1 - F(k_1)) + (1 - p^*)F(k_1)} \geq \frac{1}{b} - \frac{1}{2\sigma^2 s(0)} \frac{\sqrt{2}\sigma f(\sqrt{2}\sigma)}{(2p^* - 1)(1 - F(b)) + 1 - p^*}.$$

Since $p^*(1 - F(k_1)) + (1 - p^*)F(k_1)$ is decreasing in k_1 and $p^* \leq 1$, we get:

$$g'(k_1) \geq \frac{1}{b} - \frac{1}{\sqrt{2}\sigma s(0)} \frac{f(\sqrt{2}\sigma)}{(2p^* - 1)(1 - F(b)) + 1 - p^*} \geq \frac{1}{b} - \frac{1}{\sqrt{2}\sigma s(0)} \frac{f(\sqrt{2}\sigma)}{1 - F(b)}.$$

Therefore, if $\frac{1}{b} - \frac{1}{\sqrt{2}\sigma s(0)} \frac{f(\sqrt{2}\sigma)}{1 - F(b)} > 0$, then (S.24) has the only solution $k_1^* = 0$.
The last inequality is equivalent to

$$1 - F(b) - \frac{b}{2\sqrt{2e}\sigma} > 0$$

LHS is strictly decreasing in b, but this inequality still holds for $b^* = 1.05\sigma$:
$1 - F(1.05\sigma) - \frac{1.05}{2\sqrt{2e}} > 1 - 0.7734 - 0.2252 > 0$.
Hence, if $b \leq 1.05\sigma$, then $U_1(k_1, k_2^* = 0, p^*)$ has the only critical point $k_1^* = 0$, which is a global maximum.

As we consider the symmetric case and the mixing belongs to the principal's BR, $(0, 0, a = p^* \min(a_1, a_2) \oplus (1 - p^*) \max(a_1, a_2))$ is the only mixed equilibrium. □

In mixed equilibrium with large B, the chooses the lower offer more often implying that the offers affected by negative realizations of noise are accepted more frequently. The winner's curse causes experts to revise their markups upward, while the bonus provides an opposite incentive. Since markups must sum to zero to ensure principal indifference, mixed equilibrium requires that bonus-earning incentives prevail. In any such mixed equilibrium each expert sets a markup lower than his bias, though the expert with higher bias applies the higher markup. When the bonus increases, the principal decreases the probability of choosing the lower offer in order to keep the equilibrium.

While welfare in mixed equilibrium is difficult to characterize, we show in the next section that, in tractable cases, the principal prefers min equilibrium to mixed.

S.2 Welfare Analysis: Additional Results

First, for equally biased experts we welfare-wise compare min, max and mixed equilibria in the region where they all co-exist ($B \le B_d$) in the following proposition.

Proposition S.8. *Suppose that bonus B is exogenously given and the max equilibrium exists for $B = 0$. For any bonus B from $(0, B_d)$ the principal prefers the min equilibrium to the max and prefers the latter to the mixed one. He is indifferent between min and max equilibria for $B = 0$ and between max and mixed equilibria for $B = B_d$.*

Proof. From Propositions 6 and 7, we get: $V(k_U, k_U, L) = -(b - \frac{B}{2\sqrt{\pi}\sigma})^2 - \sigma^2 + \frac{\sigma^2}{\pi} \ge -(b + \frac{B}{2\sqrt{\pi}\sigma})^2 - \sigma^2 + \frac{\sigma^2}{\pi} = V(k_D, k_D, H)$, where the equality holds only for $B = 0$. Therefore the principal strictly prefers the min equilibrium to the max one if $B \in (0, B_d]$ and is indifferent between them if $B = 0$. Now compare the max and the mixed equilibria: $V(k_D, k_D, H) = -(b + \frac{B}{2\sqrt{\pi}\sigma})^2 - \sigma^2 + \frac{\sigma^2}{\pi} \ge -\sigma^2 = V(0, 0, p_M)$, where the equality holds only for $B = B_d$. Therefore the principal strictly prefers the max equilibrium to the mixed one if $B \in [0, B_d)$ and is indifferent between them if $B = B_d$.

□

The following proposition allows us to compare the experts' utilities in min and max equilibria, when the latter ones co-exist.

Proposition S.9. *a) Expert 1 prefers max equilibrium to min, strictly if $b_1 > b_2$;*
b) Expert 2 prefers min equilibrium to max, strictly if $b_1 > b_2$.

Expert 1 prefers max equilibrium and expert 2 prefers min equilibrium due to the following two reasons. First, the difference between the equilibrium markup and the own bias for Expert 1 is lower in max equilibrium. Moreover, as in both equilibria $k_1^* > k_2^*$, Expert 1 has more chances to be selected in max equilibrium, where the higher offer is chosen. Analogous considerations work in the opposite direction for Expert 2.

Proof of Proposition S.9. a) From Theorems 2 and 3, we have:

$$U_1^{min} = -\sigma^2 - (b_1 - b_2)^2 F(z^*) + B(1 - F(z^*)) - B(b_1 - b_2) f(z^*) + \left(\sigma^4 - \frac{B^2}{4}\right) \frac{f^2(z^*)}{F(z^*)(1 - F(z^*))};$$
$$U_1^{max} = -\sigma^2 - (b_1 - b_2)^2 (1 - F(z^*)) + BF(z^*) + B(b_1 - b_2) f(z^*) + \left(\sigma^4 - \frac{B^2}{4}\right) \frac{f^2(z^*)}{F(z^*)(1 - F(z^*))}.$$

Subtracting the two expressions,

$$U_1^{min} - U_1^{max} = B(1 - 2F(z^*)) - (b_1 - b_2)[2Bf(z^*) + (b_1 - b_2)(2F(z^*) - 1)] \le 0.$$

Hence, Expert 1 is indifferent for $b_1 = b_2$ and strictly better off in max equilibrium otherwise.

b) By symmetry we have for expert 2:

$$U_2^{min} - U_2^{max} = B(2F(z^*) - 1) + (b_1 - b_2)[2Bf(z^*) + (b_1 - b_2)(2F(z^*) - 1)] \ge 0,$$

with equality if and only if $b_1 = b_2$. □

Proposition S.10. *If $b_2 < 0$, the principal prefers simple delegation to expert 2 to mixed equilibrium whenever the latter exists.*

Proof. Recall that in any equilibrium, $sign(k_1 - b_1) = sign(k_2 - b_2)$. Since $k_1^m + k_2^m = 0$ while $b_1 + b_2 > 0$, we have $k_2^m < b_2 < 0$. The principal's utility is at most $-\sigma^2 - (k_2^m)^2 < -\sigma - b_2^2$. $\square$

For general b_2, it appears that mixed equilibrium is inferior to min equilibrium, but we have only numerical support. However, in the case of sufficiently similar biases, we can provide the result analytically. The following result shows that as one expert's bias becomes sufficiently close to the other, the principal eventually strictly prefers min equilibrium to mixed. In fact, the result makes the stronger claim that, in a formal sense, "sufficiently close" need not depend on the position of the biases.

Proposition S.11. *For fixed B and σ, let $b^* = \frac{2\sigma^2 - B}{2\sigma\sqrt{\pi}}$ denote the maximum b such that max equilibrium exists when biases are $b_1 = b_2 = b$. For any $\hat{b} < b^*$, there exists $\epsilon > 0$ such that if $x < \epsilon$, then for all $b \in [0, \hat{b}]$, the principal strictly prefers min equilibrium to mixed equilibrium when biases are $b_1 = b + x$ and $b_2 = b - x$.*

Proof of Proposition S.11. First, we show that for all $b \in [0, b^*]$ and all $x \geq 0$, mixed equilibrium exists. Note that $z^* = k_1^D - k_2^D = k_1^U = k_2^U$ is differentiable and increasing in x; to see this, recall that (10) and (11) imply (5), for which the LHS is increasing in z, while the RHS is increasing in x. The claim can then be verified by observing that the LHS of (12) is decreasing in z^* and hence decreasing in x. Since $z^*(x)$ is increasing, $(k_1^U - (b+x))(k_2^U - (b-x)) = (\sigma^2 - B/2)^2 \frac{f(z^*(x))^2}{F(z^*(x))(1-F(z^*(x))}$ is decreasing.

Next, observe from (S.5) that for $x = 0$, $z^* = 0$, and hence as $x \to 0$ $z^*(x) \to 0$. If we let k^U denote the common markup for experts in min equilibrium with $x = 0$, we also have $k_1^U, k_2^U \to k^U$. The principal's utility in mixed equilibrium is bounded above by $-\sigma^2$; we complete the proof by showing that for sufficiently small $\epsilon > 0$, for all $b \in [0, b^*]$, $V^U(b, x)$ above $-\sigma^2$ for all $x \in [0, \epsilon)$, where $V^U(b, x)$ is the value of min equilibrium:

$$V^U(b, x) = -\sigma^2 + B(k_2^U + k_2^U)f(z^*) - b^2 - x^2 + (2F(z^*) - 1)2bx + (k_1^U - (b+x))(k_2^U - (b-x)). \quad \text{(S.25)}$$

Let $V_0^U(b, x) := V^U(b, x) - B(k_2^U + k_2^U)f(z^*)$. For $x = 0$, we have $V_0^U(b, 0) = -\sigma^2 - b^2 + (k^U - b)^2$. Since in max equilibrium, $k_1 = k_2 = k_D \leq 0$, we have $b < b - k^D = k^U - b$, and this implies that $V_0^U(b, 0) \geq -\sigma^2$, with equality if and only if $b = b^*$. Now $V^U(b, x) \geq V_0^U(b, x)$ and by continuity w.r.t. x, we have the pointwise result: given b, $\lim_{x \to 0} V^U(b, x) \geq -\sigma^2$, with equality if and only if $b = b^*$. Next, consider $\hat{b} < b^*$. By continuity (in b) and compactness, $V^U(b, 0)$ is above and bounded away from $-\sigma^2$ on $[0, \hat{B}]$. Note that, individually, the terms $-x^2$, $(2F(z^*(x)) - 1)2bx$ and $(k_1^U - (b+x))(k_2^U - (b-x))$ of the expansion of $V^U(b, x)$ are monotonic in x. By Dini's Theorem[1], we have that for $b \in [0, \hat{b}]$, $V^0(b, x)$ converges uniformly in b to $V^U(b, 0)$, and thus for sufficiently small $\epsilon > 0$, $x \in [0, \epsilon)$ implies that $V^U(b, x) > -\sigma^2 \geq V^M(b, x)$. $\square$

[1]See Ok (2007).

As it was discussed in Mixed Equilibrium section, the existence of mixed equilibrium is ambiguous for large bonuses. The following proposition shows that even if mixed equilibrium exists, it is suboptimal for the principal, even when the bonus is not paid from his pocket. Therefore if the latter chooses the bonus, he doesn't enter the high bonus area.

Proposition S.12. *a) Among $\{B : B \geq B_u$ and the mixed equilibrium exists for $B\}$, the Principal is better off in $B = B_u$, even if the bonus is exogenously given.*
b) Comparing to mixed(= min) equilibrium in $B = B_u$, the Principal is better off in some left neighbourhood of $B = B_u$, when min equilibrium is played

Proof of Proposition S.12. a) The Principal's utility in mixed equilibrium is $V_m(B) = -\sigma^2 - \frac{1}{4}z_m^2(B)$. As we noted, $z_m(B)$ is increasing for $B \geq B_u$, so the maximum is achieved at point $B = B_u$.
b) Consider the mixed equilibrium for $B > B_u$. $k_1^m(B)$ and $k_2^m(B)$ are continuous functions and, as B tends to B_u, they converge to $k_1^m(B_u)$ and $k_2^m(B_u)$ correspondingly. $z^m(B_u) = z^*(B_u)$, therefore in $B = B_u$ the mixed equilibrium coincides with the min equilibrium and $V_m(B_u) = V_u(B_u)$

Now consider the min equilibrium.
$k_1(B_u) + k_2(B_u) = 0$, therefore $V_u(B_u) = V(k_1(B_u), k_2(B_u), L) = -\sigma^2 - k_2^2(B_u)$. Since $v(z) > \frac{z}{2\sigma^2}$,

$$V(k_1, k_2, L) = -\sigma^2 - k_2^2 + (k_1 + k_2)[2\sigma^2 f(k_1 - k_2) - (k_1 - k_2)(1 - F(k_1 - k_2))] \geq -\sigma^2 - k_2^2.$$

For $b_1 = b_2$ the statement can be checked directly. Otherwise, $k_1(B_u)+k_2(B_u) = 0 \implies k_2(B_u) < 0$. $k_2(B) = b_2 - \left(\frac{B}{2} - \sigma^2\right)\frac{f(z(B))}{F(z(B))}$ is continuous and decreasing for $B > 2\sigma^2$. Therefore exists a left neighbourhood of B_u, where $k_2(B_u) < k_2(B) < 0$ and in this neighbourhood:
$V_u(B) \geq -\sigma^2 - k_2^2(B) > -\sigma^2 - k_2^2(B_u) = V_u(B_u) = V_m(B_u) > V_m(B)$ for $B > B_u$. □

S.3 Proof of Lemma A.1

Let $\epsilon_1, \epsilon_2 \sim N(0, \sigma^2)$, ϵ_1 and ϵ_2 are independent;
$\xi(k_1, k_2) = \min(\epsilon_1 + k_1, \epsilon_2 + k_2)$, $\eta(k_1, k_2) = \max(\epsilon_1 + k_1, \epsilon_2 + k_2)$. Then

$$\begin{aligned}
\mathbb{E}\xi(k_1, k_2) &= -2\sigma^2 f(k_1 - k_2) + k_1(1 - F(k_1 - k_2)) + k_2 F(k_1 - k_2);\\
\mathbb{E}\eta(k_1, k_2) &= 2\sigma^2 f(k_1 - k_2) + k_1 F(k_1 - k_2) + k_2(1 - F(k_1 - k_2));\\
\mathbb{E}\xi^2(k_1, k_2) &= \sigma^2 - 2(k_1 + k_2)\sigma^2 f(k_1 - k_2) + k_1^2(1 - F(k_1 - k_2)) + k_2^2 F(k_1 - k_2);\\
\mathbb{E}\eta^2(k_1, k_2) &= \sigma^2 + 2(k_1 + k_2)\sigma^2 f(k_1 - k_2) + k_1^2 F(k_1 - k_2) + k_2^2(1 - F(k_1 - k_2)).
\end{aligned}$$

Proof. Remind that f and F denote the PDF and the CDF of $N(0, 2\sigma^2)$.
Let h and H denote the PDF and the CDF of the distribution $N(0, \sigma^2)$ correspondingly.
Start with calculation of auxiliary integrals. The first two are directly computed by substitution:

$$I_1(k_1, k_2) = \int_{\mathbf{R}} h(t - k_1)h(t - k_2)dt = \left| z = t - \frac{k_1 + k_2}{2} \right| = f(k_1 - k_2)$$
$$I_2(k_1, k_2) = \int_{\mathbf{R}} th(t - k_1)h(t - k_2)dt = \left| z = t - \frac{k_1 + k_2}{2} \right| = \frac{1}{2}(k_1 + k_2)f(k_1 - k_2)$$

The third one is calculated by parts:

$$I_3(k_1,k_2) = \int_{\mathbf{R}} H(t-k_1)dh(t-k_2) = H(t-k_1)h(t-k_2)|_{t=-\infty}^{t=\infty} - I_1(k_1,k_2) = -f(k_1-k_2)$$

The fourth one

$$I_4(k_1,k_2) = \int_{\mathbf{R}} H(t-k_1)h(t-k_2)dt = \int_{\mathbf{R}} H(t+k_2-k_1)h(t)dt := \phi(k_2-k_1)$$

$0 \le \phi(z) \le 1$ and $\phi(z)$ is well-defined. Then for any z: $\phi'(z) = I_1(z,0) = f(z)$. Integrating back, we get $\phi(z) = F(z) + C$. Integrating by parts, we receive $\phi(z) = 1 - \phi(-z)$. These equalities imply $C = 0$ and $\phi(z) = F(z)$. As a consequence, $I_4(k_1,k_2) = F(k_2-k_1) = 1 - F(k_1-k_2)$.

Finally, compute two integrals, that will be used for the calculations of RV moments:

$$\begin{aligned}
I_5(k_1,k_2) &= \int_{\mathbf{R}} tH(t-k_1)h(t-k_2)dt = -\sigma^2 I_3(k_1,k_2) + k_2 I_4(k_1,k_2) = \sigma^2 f(k_1-k_2) + k_2(1-F(k_1-k_2)) \\
I_6(k_1,k_2) &= \int_{\mathbf{R}} t^2 H(t-k_1)h(t-k_2)dt = -\sigma^2 \int_{\mathbf{R}} tH(t-k_1)dh(t-k_2) + k_2 I_5(k_1,k_2) \\
&= \sigma^2[I_2(k_1,k_2) + I_4(k_1,k_2)] + k_2 I_5(k_1,k_2) \\
&= \frac{1}{2}(k_1+3k_2)\sigma^2 f(k_1-k_2) + (k_2^2+\sigma^2)(1-F(k_1-k_2))
\end{aligned}$$

Now, return to the RVs. The PDF and the CDF of η are given by:
$F_\eta(t) = \Pr(\eta < t) = \Pr(\epsilon_1 + k_1 < t, \epsilon_2 + k_2 < t) = H(t-k_1)H(t-k_2)$,
$f_\eta(t) = H(t-k_1)h(t-k_2) + H(t-k_2)h(t-k_1)$.

Using the obtained density function, calculate the moments of η:

$$\begin{aligned}
\mathbb{E}\eta &= \int_{\mathbf{R}} tf_\eta(t)dt = I_5(k_1,k_2) + I_5(k_2,k_1) = 2\sigma^2 f(k_1-k_2) + k_1 F(k_1-k_2) + k_2(1-F(k_1-k_2)); \\
\mathbb{E}\eta^2 &= \int_{\mathbf{R}} t^2 f_\eta(t)dt = I_6(k_1,k_2) + I_6(k_2,k_1) \\
&= \sigma^2 + 2(k_1+k_2)\sigma^2 f(k_1-k_2) + k_1^2 F(k_1-k_2) + k_2^2(1-F(k_1-k_2)).
\end{aligned}$$

Finally, from the identities $\xi + \eta \equiv \epsilon_1 + k_1 + \epsilon_2 + k_2$ and $\xi^2 + \eta^2 \equiv (\epsilon_1+k_1)^2 + (\epsilon_2+k_2)^2$, we obtain the moments for ξ:

$$\begin{aligned}
\mathbb{E}\xi &= \mathbb{E}(\epsilon_1 + k_1 + \epsilon_2 + k_2 - \eta) = k_1 + k_2 - \mathbb{E}\eta \\
&= -2\sigma^2 f(k_1-k_2) + k_1(1-F(k_1-k_2)) + k_2 F(k_1-k_2); \\
\mathbb{E}\xi^2 &= \mathbb{E}[(\epsilon_1+k_1)^2 + (\epsilon_2+k_2)^2 - \eta^2] = 2\sigma^2 + k_1^2 + k_2^2 - \mathbb{E}\eta^2 \\
&= \sigma^2 - 2(k_1+k_2)\sigma^2 f(k_1-k_2) + k_1^2(1-F(k_1-k_2)) + k_2^2 F(k_1-k_2).
\end{aligned}$$

□

References

Ok, E. A. (2007). *Real analysis with economic applications*, Volume 10. Princeton University Press.