Supplementary Appendix to **Commitment-Flexibility Trade-off and Withdrawal Penalties**

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1 Additional formal results

Proposition 1 *Take any convex functions $U(\cdot)$ and $W(\cdot)$ such that the function $z(u)$ has at least one point $u_0 \in (0, y)$ with $\left|\frac{dz}{du}|_{u=u_0}\right| \geq 1$ (this would be the case, for example, if $W = U$, or if $W'(0) = \infty$ and $W(0) \neq -\infty$). Then there exists an open set of parameter values μ, θ_l, β (with θ_h found from $\mu\theta_l + (1-\mu)\theta_h = 1$) such that the optimal contract necessarily includes money-burning.*

Proof of Proposition 1. Given $U(\cdot)$ and $W(\cdot)$, the set A is fixed. Let $w = z(u)$ be the equation that determines the upper boundary of this set and let $k = \left|\frac{dz}{du}(u_0)\right| \geq 1$. By assumption that $W(0) \neq -\infty$ and convexity of A, the number $s = \frac{z(u_0)-W(0)}{U(y)-u_0} \in (k, \infty)$. For any $\beta \in \left(0, \frac{1}{s}\right) \subset (0,1)$, let $\theta_l(\beta) = \beta s$. In this case, u_0 will be the u_0 from formulation of Proposition 2 in Ambrus and Egorov (2012). We have $\mu\frac{1-\beta}{\frac{1}{\left|\frac{dz}{dx}(u_0)\right|}-\frac{\beta}{\theta_l(\beta)}} = \mu\frac{1-\beta}{\frac{1}{k}-\frac{1}{s}}$. But $s \in (k, \infty)$ and $k \geq 1$ implies $\frac{1}{k} - \frac{1}{s} \in (0,1)$, which means that inequality

$$\mu\frac{1-\beta}{\frac{1}{\left|\frac{dz}{du}|_{u=u_0}\right|} - \frac{\beta}{\theta_l}} > 1. \tag{1}$$

must hold for β sufficiently close to 0 and μ sufficiently close to 1 (and θ_l, θ_h derived by $\theta_l = \beta s$ and $\theta_h = \frac{1-\mu\theta_l}{1-\mu}$). Moreover, for μ close to 1 we will have θ_h arbitrarily high, in particular, $\theta_h > s = \frac{\theta_l(\beta)}{\beta}$. The latter implies $\beta > \frac{\theta_l}{\theta_h}$, and we have $\beta < \beta^*$ by construction, so in this case, indeed, a separating contract is optimal by Proposition 1 in Ambrus and Egorov (2012). Finally,

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since varying u_0 would not change the inequalities above, then the set of parameters β, μ, θ_l for which money-burning is optimal contains an open set. ■

Proposition 2 *If* $\beta \in \left(\frac{\theta_l}{\theta_h}, \beta^*\right)$ *(so that the optimal contract is separating but not the first-best), and* $\left|\frac{dz}{du}|_{u=u_0}\right| \geq 1$*, then the optimal contract involves money-burning. In particular, if* $z(u)$ *is such that* $\left|\frac{dz}{du}|_{u=U(0)}\right| \geq 1$ *and* $\left|\frac{dz}{du}|_{u=U(y)}\right| \leq \theta_h$ *(i.e.,* $\left|\frac{dz}{du}(u)\right| \in [1, \theta_h]$ *for all* $u \in [U(0), U(y)]$*), then for **every*** $\beta \in \left(\frac{\theta_l}{\theta_h}, \beta^*\right)$ *the optimal contract involves money-burning.*

Proof of Proposition 2. Fix u_0 and thus $\left|\frac{dz}{du}|_{u=u_0}\right| = x > 1$. Let us take $\beta_0 = \frac{\theta_l}{\theta_h} = \frac{\theta_l(1-\mu)}{1-\mu\theta_l}$ and plug it into (1). We get:

$$\begin{aligned}
\mu \frac{1-\beta_0}{\frac{1}{\left|\frac{dz}{du}|_{u=u_0}\right|} - \frac{\beta_0}{\theta_l}} - 1 &= \mu \frac{1 - \frac{\theta_l(1-\mu)}{1-\mu\theta_l}}{\frac{1}{x} - \frac{1-\mu}{1-\mu\theta_l}} - 1 \\
&= \frac{x-1}{1 - x\frac{1-\mu}{1-\mu\theta_l}} = \frac{x-1}{1-\frac{x}{\theta_h}} \geq 0,
\end{aligned}$$

because $x \leq \frac{\theta_l}{\beta} < \frac{\theta_l}{\beta_0} = \theta_h$ and $x > 1$. Notice that the left-hand side of (1) is increasing in β (again for a fixed $\left|\frac{dz}{du}|_{u=u_0}\right| = x$): indeed, we have

$$\frac{d}{d\beta}\left(\mu \frac{1-\beta}{\frac{1}{x} - \frac{\beta}{\theta_l}}\right) = x\mu\theta_l \frac{x-\theta_l}{(\theta_l - x\beta)^2} > 0,$$

as $x > 1 > \theta_l$. Consequently, for $\beta > \beta_0 = \frac{\theta_l}{\theta_h}$, condition (1) holds with strict inequality, and money-burning is optimal.

To prove the second part, it now suffices to prove that for all $\beta \in \left(\frac{\theta_l}{\theta_h}, \beta^*\right)$, $u_0 > U(0)$ (then we would have $\left|\frac{dz}{du}|_{u=u_0}\right| > 1$ and the first part would apply). But now the first-best points are $(U(y), W(0))$ for θ_h and $(U(0), W(y))$ for θ_l. Consequently, the leftmost point of the green line corresponding to $\beta < \beta^*$ that lies in A satisfies $u_0 = U(0)$, and thus the previous result is applicable. This completes the proof. ■

2 References

Ambrus, A. and G. Egorov (2012): "Commitment-flexibility trade-off and withdrawal penalties," mimeo Duke University and Northwestern University.