Supplementary Appendix to Commitment-Flexibility Trade-off and Withdrawal Penalties

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1 Additional formal results

Proposition 1 Take any convex functions $U(\cdot)$ and $W(\cdot)$ such that the function $z(u)$ has at least one point $u_0 \in (0,y)$ with $\left| \frac{dz}{du} \right|_{u=u_0} \geq 1$ (this would be the case, for example, if $W = U$, or if $W'(0) = \infty$ and $W(0) \neq -\infty$). Then there exists an open set of parameter values $\mu$, $\theta_l$, $\beta$ (with $\theta_h$ found from $\mu\theta_l + (1 - \mu)\theta_h = 1$) such that the optimal contract necessarily includes money-burning.

Proof of Proposition 1. Given $U(\cdot)$ and $W(\cdot)$, the set $A$ is fixed. Let $w = z(u)$ be the equation that determines the upper boundary of this set and let $k = \left| \frac{dz}{du} \right|_{u=u_0} \geq 1$. By assumption that $W(0) \neq -\infty$ and convexity of $A$, the number $s = \frac{z(u_0) - W(0)}{U'(y) - u_0} \in (k, \infty)$. For any $\beta \in (0, \frac{1}{k}) \subset (0,1)$, let $\theta_l(\beta) = \beta s$. In this case, $u_0$ will be the $u_0$ from formulation of Proposition 2 in Ambrus and Egorov (2012). We have

$$\mu = \frac{1 - \beta}{\left| \frac{dz}{du} \right|_{u=u_0} - \frac{\beta}{s}}.$$ 

But $s \in (k, \infty)$ and $k \geq 1$ implies $\frac{k}{s} - \frac{1}{k} \in (0,1)$, which means that inequality

$$\frac{1 - \beta}{\mu} > 1$$

must hold for $\beta$ sufficiently close to 0 and $\mu$ sufficiently close to 1 (and $\theta_l$, $\theta_h$ derived by $\theta_l = \beta s$ and $\theta_h = \frac{1 - \mu\theta_l}{1 - \mu}$). Moreover, for $\mu$ close to 1 we will have $\theta_h$ arbitarily high, in particular, $\theta_h > \frac{s - \beta}{\beta}$. The latter implies $\beta > \frac{\theta_h}{\theta_l}$ and we have $\beta < \beta^*$ by construction, so in this case, indeed, a separating contract is optimal by Proposition 1 in Ambrus and Egorov (2012). Finally,

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since varying \( u_0 \) would not change the inequalities above, then the set of parameters \( \beta, \mu, \theta_t \) for which money-burning is optimal contains an open set. \[\Box\]

**Proposition 2** If \( \beta \in \left( \frac{\theta_t}{\theta_h}, \beta^* \right) \) (so that the optimal contract is separating but not the first-best), and \( |\frac{dz}{du}|_{u=u_0} | \geq 1 \), then the optimal contract involves money-burning. In particular, if \( z(u) \) is such that \( |\frac{dz}{du}|_{u=U(0)} | \geq 1 \) and \( |\frac{dz}{du}|_{u=U(y)} | \leq \theta_h \) (i.e., \( |\frac{dz}{du}|(u) | \in [1, \theta_h] \) for all \( u \in [U(0), U(y)] \), then for every \( \beta \in \left( \frac{\theta_t}{\theta_h}, \beta^* \right) \) the optimal contract involves money-burning.

Proof of Proposition 2. Fix \( u_0 \) and thus \( |\frac{dz}{du}|_{u=u_0} | = x > 1 \). Let us take \( \beta_0 = \frac{\theta_t}{\theta_h} = \frac{\theta_t(1-\mu)}{1-\mu\theta_t} \) and plug it into (1). We get:

\[
\mu \frac{1 - \beta_0}{1 - \mu \theta_t} - 1 = \frac{1 - \theta_t(1-\mu)}{1 - \mu \theta_t} - 1 = \frac{x - 1}{1 - x \frac{1 - \mu}{1 - \mu \theta_t}} = \frac{x - 1}{1 - \frac{1 - \mu}{1 - \mu \theta_t}} \geq 0,
\]

because \( x \leq \frac{\theta_t}{\beta} < \frac{\theta_t}{\theta_0} = \theta_h \) and \( x > 1 \). Notice that the left-hand side of (1) is increasing in \( \beta \) (again for a fixed \( |\frac{dz}{du}|_{u=u_0} | = x \)): indeed, we have

\[
\frac{d}{d\beta} \left( \mu \frac{1 - \beta}{x - \beta} \right) = x \mu \theta_t \frac{x - \theta_t}{(\theta_t - x \beta)^2} > 0,
\]
as \( x > 1 > \theta_t \). Consequently, for \( \beta > \beta_0 = \frac{\theta_t}{\theta_h} \), condition (1) holds with strict inequality, and money-burning is optimal.

To prove the second part, it now suffices to prove that for all \( \beta \in \left( \frac{\theta_t}{\theta_h}, \beta^* \right) \), \( u_0 > U(0) \) (then we would have \( |\frac{dz}{du}|_{u=u_0} | > 1 \) and the first part would apply). But now the first-best points are \( (U(y), W(0)) \) for \( \theta_h \) and \( (U(0), W(y)) \) for \( \theta_t \). Consequently, the leftmost point of the green line corresponding to \( \beta < \beta^* \) that lies in \( A \) satisfies \( u_0 = U(0) \), and thus the previous result is applicable. This completes the proof. \[\Box\]

2 References