

INVESTMENTS IN SOCIAL TIES, RISK SHARING, AND INEQUALITY

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ABSTRACT. This paper investigates stable and efficient networks in the context of risk sharing, when it is costly to establish and maintain relationships that facilitate risk sharing. We find a novel trade-off between efficiency and equality: The most stable efficient networks also generate the most inequality. We then suppose that individuals can be split into groups, assuming that incomes across groups are less correlated than within a group but relationships across groups are more costly to form. The tension between efficiency and equality extends to these correlated income structures. More-central agents have stronger incentives to form across-group links, reaffirming the efficiency benefits of having highly central agents. Our results are robust to many extensions. In general, endogenously formed networks in the risk-sharing context tend to exhibit highly asymmetric structures, which can lead to stark inequalities in consumption levels.

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1. INTRODUCTION

In the context of missing formal insurance markets and limited access to lending and borrowing, incomes may be smoothed through informal risk-sharing agreements that utilize social connections. A large theoretical and empirical literature studies how well informal arrangements replace the missing markets.¹ However, the existing literature does not investigate a potential downside to these agreements: If people's network position affects the share of surplus generated by risk sharing they appropriate, social investments may be distorted and inequality may endogenously arise.²

Our starting premise is that social networks are endogenous and that their structure affects how the surplus from risk sharing is split. There is growing empirical evidence that risk-sharing networks respond to financial incentives—and that, in general, risk-sharing networks form endogenously, in a way that depends on the economic environment. See, for example, recent work by Binzel et al. (2017) and Banerjee et al. (2014b,c), which in different contexts look at how social networks respond to the introduction of financial instruments such as savings vehicles or microfinance. Our main goal is to develop a theoretical framework that can be used to think about the endogeneity of risk-sharing networks, and to aid understanding of how these networks change after certain economic interventions, or more generally after changes in the economic environment.

We provide an examination of these issues by considering a simple two-stage model. In the first stage, villagers invest in costly bilateral relationships (as in Myerson (1991) and Jackson and Wolinsky (1996)), knowing that in the second stage they will reach informal risk-sharing agreements. These agreements determine how the surplus generated by risk sharing is distributed, and they depend on the endogenous structure of the social network from the first stage. In this way we elucidate new costs associated with informal risk sharing. Once incomes have been realized, risk sharing typically reduces inequality by smoothing incomes. Nevertheless, asymmetric equilibrium networks can still generate inequality in terms of expected utilities. Agents occupying more-advantageous positions in the social network may appropriate considerably more of the benefits generated by risk sharing.

For analytical tractability, in our benchmark model we impose several specific assumptions: Agents have CARA utilities, their income realizations are jointly normal, and surplus is shared in accordance with the split-the-difference renegotiations proposed in Stole and Zwiebel (1996).³ In Section 6 we extend our main results to more general settings, dropping all of the specific assumptions above, and showing that the results are robust to the

¹An incomplete list of papers includes Rosenzweig (1988), Fafchamps (1992), Coate and Ravallion (1993), Townsend (1994), Udry (1994), Ligon, Thomas and Worrall (2002), Fafchamps and Gubert (2007), Bloch, Genicot, and Ray (2008), Angelucci and di Giorgi (2009), Jackson, Rodriguez-Barraquer and Tan (2012), Ambrus, Mobius and Szeidl (2014).

²Previous works that do consider endogenously formed networks include Bramoullé and Kranton (2007a,b) in the theoretical literature and Attanasio et al. (2012) in the experimental literature. For a related paper outside the networks framework, see Glaeser et al. (2002).

³See Brügemann, Gautier and Menzio (2018a) for non-cooperative foundations.

introduction of features missing from the benchmark model such as imperfect risk sharing, enforcement constraints, or the possibility of some coalitional deviations.

In the second stage of our model, pairs of agents who have formed a connection *commit* to a bilateral risk-sharing agreement (transfers contingent on income realizations). We investigate agreements satisfying two simple properties. First, we require agreements to be pairwise efficient, in that no pair of directly connected agents leave gains from trade on the table.⁴ Second, following Stole and Zwiebel (1996), we require the agreements to be stable insofar as they are robust to “split-the-difference” renegotiations.⁵ We show that this leads to the surplus being divided by the Myerson value,⁶ a network-specific version of the Shapley value.⁷ The transfers required to implement the agreements we identify are particularly simple. Each agent receives an equal share of aggregate realized income (as in Bramoullé and Kranton, 2007a), and on top of that state-independent transfers are made.⁸

A key implication of the Myerson value determining the division of surplus is that agents who are more centrally located, in a certain sense, receive a higher share of the surplus. Moreover, in our risk-sharing context it implies that agents receive larger payoffs from providing “bridging links” to otherwise socially distant agents than from providing local connections.⁹ Empirical evidence supports this feature of our model; see Goyal and Vega-Redondo (2007), and references therein from the organizational literature: Burt (1992), Podolny and Baron (1997), Ahuja (2000), and Mehra et al. (2001).

Our analysis considers a community that comprises different groups where all agents within each group are ex-ante identical and establishing links within groups is cheaper than across groups. We also assume that the income realizations of agents are more positively correlated within groups than across groups. Groups can represent different ethnic groups or castes in a given village, or different villages.

⁴Although we consider a model in which there is perfect bilateral risk sharing, we could easily extend the model so that some income is publicly observed, some income is private, and there is perfect risk sharing of observable income and no risk sharing of unobservable income. This would be consistent with the theoretical predictions of Cole and Kocherlakota (2001) and the empirical findings of Kinnan (2011). In the CARA utilities setting, such unobserved income outside the scope of the risk-sharing arrangement does not affect our results.

⁵Stole and Zwiebel (1996) model bargaining between many employees and an employer. This scenario can be represented by a star network with the employer at the center. We extend their approach to general network structures.

⁶For non-cooperative foundations for the Myerson value, see Brügemann, Gautier and Menzio (2018b), Fontenay and Gans (2014), and Navarro and Perea (2013). Slikker (2007) also provides non-cooperative foundations, although the game analyzed is not decentralized; offers are made at the coalitional level.

⁷The Myerson value is also often assumed in social networks contexts, on normative grounds, as a fair allocation; see a related discussion on pp. 422–425 of Jackson (2010).

⁸For investigations of the division of surplus in social networks in other contexts, see Calvo-Armengol (2001, 2003), Corominas-Bosch (2004), Manea (2011), Kets, Iyengar, Sethi and Bowles (2011), and Elliott and Nava (2019).

⁹More precisely, in Section 4 we introduce the concept of Myerson distance to capture the social distance between agents in the network, and show that a pair of agents’ payoffs from forming a relationship are increasing in this measure.

We first consider the case of homogeneous agents, that is, when there is only one group. Using the inclusion–exclusion principle from combinatorics,¹⁰ we develop a new metric, which we call the Myerson distance, to describe how far apart two agents located in a network are. Using this distance, we provide a complete characterization of stable networks.

We find that even when agents are ex-ante identical, efficient networks might be stable only if they are extremely asymmetric, thereby identifying a novel trade-off between efficiency and equality. Among all possible efficient network structures, we find that the most stable one (in the sense of being stable for the largest set of parameter values) results in the most unequal division of surplus (for any inequality measure in the Atkinson class). Conversely, the least stable efficient network entails the most equal division of surplus among all efficient networks. Although agents are ex-ante identical, efficiency considerations push the structure of social connections toward asymmetric outcomes that elevate certain individuals. Socially central individuals emerge endogenously from risk-sharing considerations alone. The intuition for this result is that the star network minimizes the distance between periphery agents and hence provides the least incentives for them to establish non-essential—and therefore socially inefficient—extra links.

Turning attention to the case of multiple groups, we find that across-group underinvestment (no connection between two groups even if it would be socially efficient) becomes an issue when the cost of maintaining links across groups is sufficiently high.¹¹ The reason is that the agents who establish the first connection across groups receive less than the social surplus generated by the link, providing positive externalities for peers in their groups. To consider which agents are best incentivized to provide across-group links, we introduce a new measure of network centrality which we term Myerson centrality. Agents who are more central in this sense have better incentives to provide across-group links. This provides a second force that pushes some agents within a group to be more central than others. For example, we show that with two groups the most stable efficient network structure involves stars within groups, connected by their centers. This reinforces the trade-off between efficiency and equality in the many-groups context.

Beyond this central takeaway, our results also suggest that within homogeneous groups the likely source of inefficiency is overinvestment, as agents might spend too much time building social capital in order to occupy more-central positions in the network. On the other hand, across groups (communities) it is more likely that underinvestment inefficiency will occur, as the agents who establish the first connections do not receive the full social benefits of the link, and they exert a positive externality on other agents in their groups. Empirical work suggests that both these types of inefficiencies in investments into social capital that arise in our model are possible, in different contexts. Austen-Smith and Fryer (2005) cites numerous references from sociology and anthropology, suggesting that members of poor communities

¹⁰See Chapter 10 in van Lint and Wilson (2001).

¹¹While across-group overinvestment remains possible, the main concern when across-group link costs are relatively high is underinvestment.

allocate inefficiently large amounts of time to activities maintaining social ties instead of to productive activities. In contrast, Feigenberg et al. (2013) find evidence in a microfinance setting that it is relatively easy to experimentally intervene and create social ties among people that yield substantial benefits, suggesting underinvestment in social relationships.

We provide several generalizations of the model that show that the main insights from the benchmark model are robust, and identify additional channels strengthening the results. When bilateral transactions are costly and hence risk sharing is imperfect, efficiency requires shorter path-lengths between agents, adding an additional force for the emergence of central agents with direct connections to many others in the society. When relationships may fail such that transfers between the affected agents become impossible, redundancy needs to be built into the network, and the most robust and cost-effective way to do that is to have a few highly central agents. When risk-sharing agreements between two agents can be enforced only if they have a common friend (in the spirit of Jackson et al. (2012) and Renou and Tomala (2012)), the efficient network structure most robust to a simple form of coalitional deviations is a star structure of triangles, with one highly connected agent in the center. We also show that if there are some exogenously given links, such as family relationships, that partition a society into various components, then the most stable efficient network will again require a star structure of these components, with one central component connecting to all others. Finally, using the Moore bound from graph theory, we show that there is a basic tension between efficiency and equality/stability that does not depend on the fine details of the modeling of the economic interactions.

Among the theoretical studies on social networks and informal risk sharing that are most related to ours are Bramoullé and Kranton (2007a,b), Bloch et al. (2008), Jackson et al. (2012), Billand et al. (2012), Ali and Miller (2013, 2016), Ambrus et al. (2014), and Ambrus Gao and Milan (2016). Many of these papers focus on the enforcement issues we mainly abstract from, and investigate how social capital can be used to sustain cooperation for lower discount factors than would otherwise be possible. We take a complementary approach and instead focus on the distribution of surplus and the incentives this creates for social investments. One way of viewing our approach is an assumption on the discount factor in a dynamic version of our model. As long as the discount factor is high enough, our equilibrium agreements satisfy the necessary incentive-compatibility constraints to be able to be enforced in equilibrium of the dynamic game. And for a range of discount factors below this threshold, our results on enforcement in our extensions section apply.

Among the aforementioned papers, Bramoullé and Kranton (2007a,b) and Billand et al. (2012) investigate endogenously and costly formed networks. Bramoullé and Kranton's (2007a,b) model assumes that the surplus on a connected income component is equally distributed, independently of the network structure. This rules out the possibility of overinvestment or inequality, and leads to different types of stable networks than in our model. Instead of assuming optimal risk-sharing arrangements, Billand et al. (2012) assume an exogenously

given social norm which prescribes that high-income agents transfer a fixed amount of resources to all low-income neighbors. This again leads to very different predictions regarding the types of networks that form in equilibrium.

More generally, understanding the structure of endogenously formed networks is important. Establishing and maintaining social connections (relationships) is costly, in terms of time and other resources. However, on top of direct consumption utility, such links can yield many economic benefits. Papers studying the structure of formed networks in different contexts include Jackson and Wolinsky (1996), Bala and Goyal (2000), Kranton and Minehart (2001), Hojman and Szeidl (2008), and Elliott (2015). Although we study a specific problem tailored to risk sharing in villages, the general structure of our problem is relevant to other applications.¹²

The remainder of the paper is organized as follows. Section 2 describes risk sharing on a fixed network. In Section 3 we introduce a game of network formation with costly link formation. We focus on the structure of networks formed within a single group in Section 4 and then turn to networks spanning multiple groups in Section 5. Section 6 provides generalizations and extensions of our benchmark model, while Section 7 concludes.

2. PRELIMINARIES AND RISK SHARING ON A FIXED NETWORK

To study social investments and the structure of formed networks we first need to specify what risk-sharing arrangements take place once the network is formed. Below we introduce an economy in which agents face random income realizations, introduce some basic network terminology, and discuss risk-sharing arrangements for a given network.

2.1. The socio-economic environment. We denote the set of agents in our model by \mathbf{N} , and assume that it is partitioned into a set of groups, \mathbf{M} . We let $G : \mathbf{N} \rightarrow \mathbf{M}$ be a function that assigns each agent to a group, that is, if $G(i) = g$ then agent i is in group g . One interpretation of the group partitioning is that \mathbf{N} represents individuals in a region (such as a district or subdistrict), and the groups correspond to different villages in the region. Another possible interpretation is that \mathbf{N} represents individuals in a village, and the groups correspond to different castes.

Agents in \mathbf{N} face uncertain income realizations, with expected value μ and variance σ^2 for each agent. We assume that the correlation coefficient between the incomes of any two agents within the same group is ρ_w , while between the incomes of any two agents not in the same group it is $\rho_a < \rho_w$.¹³ That is, we assume that incomes are more positively correlated

¹²For a different and more specific application, suppose researchers can collaborate on a project. Each researcher brings something different and positive to the value of the collaboration, so that the value of the collaboration is increasing in the set of agents involved. Collaboration is possible only when it takes place among agents who are directly connected to another collaborator and surplus is split according to the Myerson value (as in our work, where the split of surplus is motivated by robustness to split-the-difference renegotiations). Such a setting fits into our framework.

¹³It is well known that for a vector of random variables, not all combinations of correlations are possible. We implicitly assume that our parameters are such that the resulting correlation matrix is positive semidefinite.

within groups than across groups, so that all else equal, social connections across groups have a higher potential for risk sharing.

Although we introduce the possibility of correlated incomes in a fairly stylized way, our paper is one of the first to permit differently correlated incomes between different pairs of agents. Such correlations are central to the effectiveness of risk-sharing arrangements, as shown below.

We refer to possible realizations of the vector of incomes as *states*, and denote a generic state by ω . We let $y_i(\omega)$ denote the income realization of agent i in state ω . Agents can redistribute realized incomes, hence their consumption levels can differ from their realized incomes.

In our benchmark model we make a number of simplifying assumptions. First, we assume that all agents have constant absolute risk aversion (CARA) utility functions

$$(1) \quad v(c_i) = -\frac{1}{\lambda} e^{-\lambda c_i},$$

where c_i is agent i 's consumption and $\lambda > 0$ is the coefficient of absolute risk aversion. Second, we assume that incomes are jointly normally distributed.¹⁴ The assumption of CARA utilities, together with jointly normally distributed incomes, greatly enhances the tractability of our model: As we show below, it leads to a transferable utility environment in which the implemented risk-sharing arrangements are relatively simple. This utility formulation can be considered a theoretical benchmark case with no income effects. We generalize the theory, relaxing these and other assumptions, in Section 6.

2.2. Basic network terminology. Before proceeding, we introduce some standard terminology from network theory. A social network L is an undirected network, with node set \mathbf{N} , where the nodes represent the agents, and links representing social connections between agents. Abusing notation, we also let L denote the set of links in the network. We will refer to the agents linked to agent i (i.e., the agents in $\mathbf{N}(i; L) := \{j : l_{ij} \in L\} \subset \mathbf{N}$) as i 's *neighbors*. Where there should be no confusion, we abuse notation by writing $\mathbf{N}(i)$ instead of $\mathbf{N}(i; L)$. The *degree centrality* of an agent is simply the number of neighbors she has (i.e., the cardinality of $\mathbf{N}(i; L)$). The set comprising an agent's neighbors can be partitioned according to the groups they belong to. Let $\mathbf{N}_g(i; L)$ be the set of i 's neighbors on network L from group g . A *walk* is a sequence of agents $\{i, k, k', \dots, k'', j\}$ such that every pair of adjacent agents in the sequence is linked. A *path* is a walk in which all agents are different. The *path length* of a path is the number of agents in the path.

We will sometimes refer to subsets of agents $\mathbf{S} \subseteq \mathbf{N}$ and denote the subnetworks they generate by $L(\mathbf{S}) := \{l_{ij} \in L : i, j \in \mathbf{S}\}$. A subset of agents $\mathbf{S} \subseteq \mathbf{N}$ is *path connected* on L if, for each pair of agents $i, j \in \mathbf{S}$, there exists a path connecting i and j . For any network,

¹⁴This specification implies that we cannot impose a lower bound on the set of feasible consumption levels. As we show below, our framework readily generalizes to arbitrary income distributions, but the assumption of normally distributed shocks simplifies the analysis considerably.

there is a unique partition of \mathbf{N} such that there are no links between agents in different partition elements but all agents within a partition element are path-connected. We refer to these partition elements as *network components*. A *shortest path* between two path-connected agents i, j is a path connecting i and j with a length no larger than any other path between them. The *diameter* of a network component $C \subset L$ is $d(C)$, the maximum value—taken over all pairs of agents in C —of the length of a shortest path. A network component is a *tree* when there is a unique path between any two agents in the component. A *line network* is the unique (tree) network, up to a relabeling of agents, in which there is a path from one (end) agent to the other (end) agent that passes through all other agents. A *star network* is the unique tree network, up to a relabeling of agents, in which one (center) agent is connected to all other agents.

2.3. Risk-Sharing Agreements. In our benchmark model, we assume that income can be directly shared between agents $i, j \in \mathbf{N}$ if and only if they are connected, that is, $l_{ij} \in L$.¹⁵ We let agents' income realizations be publicly observed within their network component, so agents can make transfer arrangements contingent on it. We consider this environment, with perfectly observable incomes within a component, as a benchmark because it is a relatively good description of village societies in which people closely monitor one another. It is also straightforward to extend the model so that some income is publicly observed (and shared) while the remaining income is privately observed (and never shared). Results are very similar for this more general setting.¹⁶

Formally, a risk-sharing agreement on a network L specifies a transfer $t_{ij}(\omega, L) = -t_{ji}(\omega, L)$ between neighboring agents i, j for every possible state ω . Abusing notation where there should be no confusion, we sometimes drop the second argument and write $t_{ij}(\omega)$ instead of $t_{ij}(\omega, L)$. The interpretation is that in state ω agent i is supposed to transfer $t_{ij}(\omega)$ units of consumption to agent j if $t_{ij}(\omega) > 0$, and to receive this amount from agent j if $t_{ij}(\omega) < 0$. Given a transfer arrangement between neighboring agents, agent i 's consumption in state ω is $c_i(\omega) = y_i(\omega) - \sum_{j \in \mathbf{N}(i)} t_{ij}(\omega)$. It is straightforward to show that state-contingent consumption plans $(c_i(\cdot))_{i \in \mathbf{N}}$ are feasible, that is, they can be achieved by bilateral transfers between neighboring agents, if and only if for each component C , $\sum_{i \in S} c_i(\omega) = \sum_{i \in S} y_i(\omega)$ for every state ω , where S is the set of agents in component C .

A basic assumption we make in our model is that given all other risk-sharing arrangements, an agreement reached by linked agents i, j must leave no gains from trade on the table. There

¹⁵In Section 6, motivated by the literature on self-enforcing risk-sharing agreements, we relax this assumption and instead consider the possibility that a link can be used for risk sharing if and only if it is supported (i.e., the two individuals have a friend in common).

¹⁶Kinnan (2011) finds evidence that hidden income can explain imperfect risk sharing in Thai villages relative to the enforceability and moral hazard problems we are abstracting from. Cole and Kocherlakota (2001) show that when individuals can privately store income, state-contingent transfers are not possible and risk sharing is limited to borrowing and lending.

must be no other agreement that can make both i and j strictly better off holding fixed the agreements of other players. We call such transfers *pairwise efficient*.¹⁷

By the well-known Borch rule (see Borch (1962), Wilson (1968)), a necessary and sufficient condition for this property is that for all neighboring agents i, j ,

$$(2) \quad \left(\frac{\partial v_i(c_i(\omega))}{\partial c_i(\omega)} \right) / \left(\frac{\partial v_j(c_j(\omega))}{\partial c_j(\omega)} \right) = \left(\frac{\partial v_i(c_i(\omega'))}{\partial c_i(\omega')} \right) / \left(\frac{\partial v_j(c_j(\omega'))}{\partial c_j(\omega')} \right)$$

for every pair of states ω and ω' . But if this holds for all neighboring agents i, j then the same condition must hold for all pairs of agents on the same component of L , regardless of whether they are directly or indirectly connected. Thus pairwise-efficient risk-sharing arrangements are equivalent to Pareto-efficient agreements at the component level. For this reason, below we establish some important properties of Pareto-efficient risk-sharing arrangements on components.

Proposition 1 shows that the CARA utilities framework has the convenient property that expected utilities are transferable, in the sense defined by Bergstrom and Varian (1985). This can be used to show that ex-ante Pareto efficiency is equivalent to minimizing the sum of the variances, and it is achieved by agreements that in every state split the sum of the incomes on each network component equally among the members and then adjust these shares by state-independent transfers. The latter determine the division of the surplus created by the risk-sharing agreement. We emphasize that this result does not require any assumption on the distribution of incomes, only that agents have CARA utilities.

Proposition 1. *For CARA utility functions, certainty-equivalent units of consumption are transferable across agents, and if $L(\mathbf{S})$ is a network component the Pareto frontier of ex-ante risk-sharing agreements among agents in \mathbf{S} is represented by a simplex in the space of certainty-equivalent consumption. The ex-ante Pareto-efficient risk-sharing agreements for agents in \mathbf{S} are those that satisfy*

$$\min \sum_{i \in \mathbf{S}} \text{Var}(c_i) \quad \text{subject to} \quad \sum_{i \in \mathbf{S}} c_i(\omega) = \sum_{i \in \mathbf{S}} y_i(\omega) \quad \text{for every state } \omega,$$

and they consist of agreements of the form

$$c_i(\omega) = \frac{1}{|\mathbf{S}|} \sum_{k \in \mathbf{S}} y_k(\omega) + \tau_i \quad \text{for every } i \in \mathbf{S} \text{ and state } \omega,$$

where $\tau_i \in \mathbb{R}$ is a state-independent transfer made to i and $\sum_{k \in \mathbf{S}} \tau_k = 0$.

Proposition 1, which is proved in Appendix I, implies that the total surplus generated by efficient risk-sharing arrangements is an increasing function of the reduction in aggregate consumption variance (the sum of consumption variances). For a general distribution of

¹⁷More formally, transfers $\{t_{ij}(\omega, L)\}_{\omega \in \Omega, i, j: l_{ij} \in L}$ are pairwise efficient for a network L if there is no pair of agents $i, j: l_{ij} \in L$, and there are no alternative transfers $\{t'_{kl}(\omega, L)\}_{\omega \in \Omega, k, l: l_{kl} \in L}$ such that $t'_{kl}(\omega, L) = t_{kl}(\omega, L)$ for all k, l with $|\{k, l\} \cap \{i, j\}| \leq 1$, and all $\omega \in \Omega$, that give both i and j strictly higher expected utility.

shocks, this function can be complicated. However, if shocks are jointly normally distributed, then $c_i = \frac{1}{|\mathbf{S}|} \sum_{k \in \mathbf{S}} y_k + \tau_i$ is also normally distributed and $E(v(c_i)) = E(c_i) - \frac{\lambda}{2} \text{Var}(c_i)$.¹⁸ Thus in this case the total social surplus generated by efficient risk-sharing agreements is proportional to the aggregate consumption variance reduction. This greatly simplifies the computation of surpluses in the analysis below.

We use $TS(L)$ to denote the expected total surplus generated by an ex-ante Pareto-efficient risk-sharing agreement on network L , relative to agents consuming in autarky:

$$(3) \quad TS(L) := CE\left(\Delta \text{Var}(L, \emptyset)\right),$$

where, for $L' \subset L$, $\Delta \text{Var}(L, L')$ is the additional variance reduction obtained by efficient risk sharing on network L instead of L' , and $CE(\cdot)$ denotes the certainty-equivalent value of a variance reduction.

For a network L that consists of a single component, if all agents are from the same group, then as there are CARA utility functions and normally distributed incomes,

$$(4) \quad TS(L) = CE\left(\Delta \text{Var}(L, \emptyset)\right) = \frac{\lambda}{2} \left(\Delta \text{Var}(L, \emptyset)\right) = \frac{\lambda}{2} (n-1) \sigma^2 (1 - \rho_w) = (n-1)V,$$

where $V := \frac{\lambda}{2} \sigma^2 (1 - \rho_w)$.

2.4. Division of Surplus. The assumption that neighboring agents make pairwise-efficient risk-sharing agreements pins down agreements up to state-independent transfers between neighboring agents but does not constrain the state-independent transfers (and hence the division of surplus) in any way. In our benchmark model, to determine these transfers we follow the approach in Stole and Zwiebel (1996) and require agreements to be robust to split-the-difference renegotiations. This implies that the transfers are set in such a way that the incremental benefit that the link provides to the two agents is split equally between them. This can be interpreted as a social norm. For a detailed motivation of this assumption, see Stole and Zwiebel (1996), and for non-cooperative microfoundations see Brügemann et al. (2018a). In Section 6 we consider a much larger set of risk-sharing agreements and show that our main results still hold.

Splitting the incremental benefits of a risk-sharing link equally between two agents i, j requires calculating the expected payoffs those agents would receive if they did not have an agreement. Thus to find the risk-sharing agreement agents i and j reach on L we have to consider what agreements would prevail on the network without l_{ij} , and so on. This results in a recursive system of conditions.

More formally, for a network L a contingent transfer scheme

$$(5) \quad \mathcal{T}(L) := \{t_{ij}(\omega, L')\}_{\omega \in \Omega, L' \subseteq L, i, j: l_{ij} \in L}$$

¹⁸See, for example, Arrow (1965).

specifies all transfers made in all subnetworks of L in all states of the world. The expected utility of agent i on a network $L' \subseteq L$ given a contingent transfer scheme $\mathcal{T}(L)$ is denoted by $u_i(L', \mathcal{T}(L))$. Where there should be no confusion, we will abuse notation and drop the second argument.

For any network L , the expected utility vector $(u_1, \dots, u_{|\mathbf{N}|})$ is *robust to split-the-difference renegotiations* if there is a contingent transfer scheme $\mathcal{T}(L)$ such that $u_i = u_i(L, \mathcal{T}(L))$ for every $i \in \mathbf{N}$ and the following conditions hold:

- (i) $u_i(L') - u_i(L' \setminus \{l_{ij}\}) = u_j(L') - u_j(L' \setminus \{l_{ij}\})$ for all $L' \subseteq L$ and all i, j such that $l_{ij} \in L'$;
- (ii) transfers $\{t_{ij}(\omega, L')\}_{\omega \in \Omega, i, j: l_{ij} \in L'}$ are pairwise efficient for all $L' \subseteq L$ and all i, j such that $l_{ij} \in L'$.

Suppose that all agents are from the same group, we have CARA utilities, incomes are normally distributed, and we want to find payoffs robust to split-the-difference renegotiations for the line network shown in Figure 1a. A first necessary condition is that agents 1 and 2 benefit equally from their link, so that $u_1(L) - u_1(L \setminus \{l_{12}\}) = u_2(L) - u_2(L \setminus \{l_{12}\})$. But in order to ensure that this condition is satisfied, we need to know $u_1(L \setminus \{l_{12}\})$ and $u_2(L \setminus \{l_{12}\})$. Normalizing the autarky utility of all agents to 0, without the link l_{12} agent 1 is isolated, so $u_1(L \setminus \{l_{12}\}) = 0$. However, to find $u_2(L \setminus \{l_{12}\})$, we need to find payoffs for the three-node network in Figure 1b. For this network, robustness to split-the-difference renegotiations requires that $u_2(L \setminus \{l_{12}\}) - u_2(L \setminus \{l_{12}, l_{23}\}) = u_3(L \setminus \{l_{12}\}) - u_3(L \setminus \{l_{12}, l_{23}\})$. While $u_2(L \setminus \{l_{12}, l_{23}\}) = 0$, we need to consider the two-node network shown in Figure 1c to find $u_3(L \setminus \{l_{12}, l_{23}\})$. For this network, payoffs must satisfy $u_3(L \setminus \{l_{12}, l_{23}\}) - u_3(L \setminus \{l_{12}, l_{23}, l_{34}\}) = u_4(L \setminus \{l_{12}, l_{23}\}) - u_4(L \setminus \{l_{12}, l_{23}, l_{34}\})$. As $u_3(L \setminus \{l_{12}, l_{23}, l_{34}\}) = u_4(L \setminus \{l_{12}, l_{23}, l_{34}\}) = 0$, the above condition simplifies to $u_3(L \setminus \{l_{12}, l_{23}\}) = u_4(L \setminus \{l_{12}, l_{23}\}) = V/2$, where the last equality follows from pairwise efficiency. Considering the three-node network again, we now have the condition $u_2(L \setminus \{l_{12}\}) = u_3(L \setminus \{l_{12}\}) - V/2$. As the link l_{23} generates an incremental surplus of V to be split between agents 2 and 3, pairwise efficiency implies that $u_2(L \setminus \{l_{12}\}) = V/2$ and $u_3(L \setminus \{l_{12}\}) = V$. Finally, returning to the line network, we now have $u_1(L) = u_2(L) - V/2$. As the link l_{12} generates incremental surplus of V , $u_1(L) = V/2$ and $u_2(L) = V$.¹⁹

Below we show that the requirement of robustness to split-the-difference renegotiations implies that the total surplus created by the risk-sharing agreement is divided among agents according to the Myerson value (Myerson 1977, 1980). The Myerson value is a cooperative solution concept defined in transferable utility environments that is a network-specific version

¹⁹This argument outlines only why the payoffs $u_1(L) = V/2$ and $u_2(L) = V$ are necessary for robustness to split-the-difference renegotiations. By considering all other subnetworks, it can be shown that the payoffs $u_1(L) = u_4(L) = V/2$ and $u_2(L) = u_3(L) = V$ are the unique payoffs that are robust to split-the-difference renegotiations.

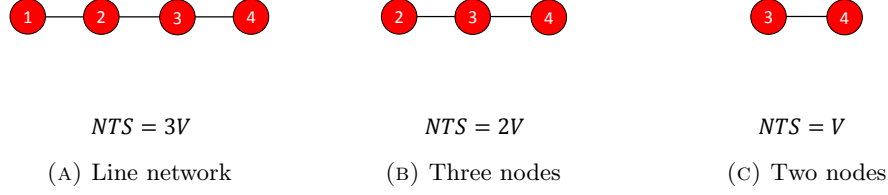


FIGURE 1. To find (gross) expected utilities that are robust to split-the-difference renegotiations on the (formed) line network shown in (A) we need to consider the expected utilities that would be obtained on all subnetworks.

of the Shapley value. The basic idea behind it is the same as for the Shapley value.²⁰ For any order of arrivals of the players, the incremental contribution of agent i to the total surplus can be derived as the difference between the total surpluses generated by subnetwork $L(\mathbf{S})$ and subnetwork $L(\mathbf{S} \setminus \{i\})$ if the agents in $\mathbf{S} \setminus \{i\}$ arrive before i . It is easy to see that, for any arrival order, the total surplus generated by L gets exactly allocated to the set of all agents. The Myerson value then allocates the average incremental contribution of a player to the total surplus, taken over all possible orders of arrivals (permutations) of the players, as that player's share of the total surplus. Thus agent i 's Myerson value is²¹

$$(6) \quad MV_i(L) := \sum_{\mathbf{S} \subseteq \mathbf{N}} \frac{(|\mathbf{S}| - 1)! (|\mathbf{N}| - |\mathbf{S}|)!}{|\mathbf{N}|!} \left(TS(L(\mathbf{S})) - TS(L(\mathbf{S} \setminus \{i\})) \right).$$

Proposition 2. *For any network L , any risk-sharing agreement that is robust to split-the-difference renegotiations yields expected payoffs to agents equal to their Myerson values: $u_i(L) = MV_i(L)$.*

Proof. Theorem 1 of Myerson (1980) states that there is a unique rule for allocating surplus for all subnetworks of L that satisfies the requirements of efficiency at the component level (note that this is an implicit requirement in Myerson's definition of an allocation rule) and what Myerson (1980) defines as the equal-gains principle. Moreover, the expected payoff the above rule allocates to player i is MV_i . Condition (i) in our definition of robustness to split-the-difference renegotiations is equivalent to the equal-gains principle as defined in Myerson (1980). Theorem 1 of Wilson (1968) implies that efficiency at the component level is equivalent to pairwise efficiency between neighboring agents, which is condition (ii) in

²⁰We therefore follow Hart and Moore (1990), among others, in using the Shapley value to study investment decisions.

²¹Our assumption that there is perfect risk sharing among path-connected agents ensures that a coalition of path-connected agents generates the same surplus regardless of the exact network structure connecting them. This means that we are in the communication game world originally envisaged by Myerson. We do not require the generalization of the Myerson value to network games proposed in Jackson and Wolinsky (1996), which somewhat confusingly is also commonly referred to as the Myerson value. See Ambrus et al. (2016) for a model of informal risk sharing in which the exact shape of the network matters in terms of the surplus that agents can attain.

our definition of robustness to split-the-difference renegotiations. The result then follows immediately from Theorem 1 of Myerson (1980). \square

Proposition 2 is a direct implication of Myerson’s axiomatization of the value. A special case of Proposition 2 is Theorem 1 of Stole and Zwiebel (1996), which in effect restricts attention to a star network.²² Our contribution is to point out that their connection between robustness to split-the-difference renegotiations and the Shapley value can be extended to apply to all networks.

The above result shows that any decentralized negotiation procedure between neighboring agents that satisfies two natural properties (not leaving surplus on the table, and robustness to split-the-difference negotiations) leads to the total surplus created by risk sharing divided according to the Myerson value, and to state-independent transfers between neighboring agents that implement this surplus division. Thus from now on we assume that all agents expect the surplus to be divided according to the Myerson value implied by the network that eventually forms.

Although we followed a decentralized approach to get to the implication that surplus is divided by the Myerson value, we note that on normative grounds such a division is also cogent in contexts in which there is a centralized community-level negotiation over the division of surplus. This is because the Myerson value is a formal way of defining the fair share of an individual from the social surplus as her average incremental contribution to the total social surplus (where the average is taken across all possible orders of arrival of different players, in the spirit of the Shapley value).

3. INVESTING IN SOCIAL RELATIONSHIPS: BENCHMARK CASE

Having defined how formed networks map into risk-sharing arrangements, we can now consider agents’ incentives to make investments into social capital, which we think of as the set of relationships that enable risk sharing. We begin by providing the overall framework for the analysis. Then we look at a special case of our model in which there is a single group. Building on these results, we then consider the multiple-group case.

In this section we formalize a game of network formation in which establishing links is costly, define efficient networks, and identify different types of investment inefficiency.

We consider a two-period model in which in period 1 all agents simultaneously choose which other agents they would like to form links with, and in period 2 agents agree upon the ex-ante Pareto-efficient risk-sharing agreement specified in the previous section (i.e., the total surplus from risk sharing is distributed according to the Myerson value) for the network formed in the first period.²³

²²Relative to Myerson’s axiomatization, Stole and Zwiebel (1996) generate the key system of equations by considering robustness to split-the-difference renegotiations as we describe above, while Myerson wrote down the system of equations based only on fairness considerations.

²³For a complementary treatment of endogenously formed networks when surplus is split according to the Myerson value, see Pin (2011).

Implicit in our formulation of the timing of the game is the view that relationships are formed over a longer time horizon than that in which agreements are reached about risk sharing. By the time such agreements are being negotiated, the network structure is fixed and investments into forming social relationships are sunk. In addition, as mentioned in the introduction, the second-stage agreements can be viewed as a reduced-form treatment of a dynamic game with many state realizations—as long as the discount factor is high enough, our agreements will satisfy the required incentive compatibility constraints for an equilibrium. We also relax our assumptions on enforcement in Section 6.

In period 1 the solution concept we apply to identify which networks form is pairwise stability. The collection of links formed is social network L , and agent i pays a cost $\kappa_w > 0$ for each link she has to someone in her own group, and $\kappa_a > \kappa_w$ for each link she has to someone from a different group. Normalizing the utility from autarky to 0, we abuse notation²⁴ and let agent i 's *net* expected utility if network L forms be

$$(7) \quad u_i(L) = MV_i(L) - |\mathbf{N}_{G(i)}(i; L)|\kappa_w - \left(|\mathbf{N}(i; L)| - |\mathbf{N}_{G(i)}(i; L)|\right)\kappa_a.$$

A network L is *pairwise stable* with respect to expected utilities $\{u_i(L)\}_{i \in \mathbf{N}}$ if and only if for all $i, j \in \mathbf{N}$, (i) if $l_{ij} \in L$ then $u_i(L) - u_i(L \setminus \{l_{ij}\}) \geq 0$ and $u_j(L) - u_j(L \setminus \{l_{ij}\}) \geq 0$; and (ii) if $l_{ij} \notin L$ then $u_i(L \cup \{l_{ij}\}) - u_i(L) > 0$ implies that $u_j(L \cup \{l_{ij}\}) - u_j(L) < 0$. In words, pairwise stability requires that no two players can both strictly benefit by establishing an extra link with each other, and no player can benefit by unilaterally deleting one of her links. From now on we will use the terms pairwise stable and stable interchangeably.

Existence of a pairwise-stable network in our model follows from a result in Jackson (2003) which states that whenever payoffs in a simultaneous-move network formation game are determined based on the Myerson value, there exists a pairwise-stable network.

Our specification assumes that two agents forming a link have to pay the same cost for establishing the link. However, the set of stable networks would remain unchanged if we allowed the agents to share the total costs of establishing a link arbitrarily.²⁵ This is because for any link, the Myerson value rewards the two agents establishing the link symmetrically. Thus the agents can find a split of the link-formation cost such that establishing the link is profitable for both of them if and only if it is profitable for both of them to form the link with an equal split of the cost. Given this, we stick with the simpler model with exogenously given costs.

A network L is *efficient* when there is no other network L' —and no risk-sharing agreement on L' —that can make everyone at least as well off as they were on L and someone strictly

²⁴In the previous section, when investments had already been sunk we used $u_i(L)$ to denote i 's expected payoff before link-formation costs.

²⁵More precisely, we could allow agents to propose a division of the costs of establishing each link as well as indicating whom they would like to link to, and a link would then form between a given pair of agents only if they both indicate each other and they propose the same split of the cost. A network would then be stable if it is a Nash equilibrium of this expanded network formation game and there is no new link $l_{ij} \notin L$ —and some split of the cost of forming this link—that would make both i and j strictly better off if formed.

better off. Let $|L_w|$ be the number of within-group links, and let $|L_a|$ be the number of across-group links. As expected utility is transferable in certainty-equivalent units, efficient networks must maximize the net total surplus $NTS(L)$:

$$(8) \quad NTS(L) := TS(L) - 2|L_w|\kappa_w - 2|L_a|\kappa_a,$$

Clearly, two necessary conditions for a network to be efficient are that the removal of a set of links does not increase $NTS(L)$ and that the addition of a set of links does not increase $NTS(L)$. If there exists a set of links the removal of which increases $NTS(L)$, we will say there is *overinvestment* inefficiency. If there exists a set of links the addition of which increases $NTS(L)$, we will say there is *underinvestment* inefficiency.²⁶ A network is *robust to underinvestment* if there is no underinvestment inefficiency and no agent can strictly benefit from deleting a link that would result in underinvestment inefficiency. A network is *robust to overinvestment* if there is no overinvestment inefficiency and no pair of agents i, j can both strictly benefit from creating the link l_{ij} .

We will say that a link l_{ij} is *essential* if after its removal i and j are no longer path connected while it is *superfluous* if after its removal i and j are still path connected.

Remark 3. *Preventing overinvestment requires that all links be essential. Superfluous links create no social surplus and are costly. In all efficient networks, therefore, every component must be a tree.*

Real-world networks among villagers are a long way from being trees. If our model perfectly captured network formation, Remark 3 would imply that there is substantial overinvestment. However, our model is stylized, and this result needs to be applied with caution. For example, while there may be overinvestment, our assumption that all links are costly to form is unlikely to hold. Family ties or the time villagers spend working together might permit relationships to be formed without any additional investment. We discuss in Section 6.4 how what we view as the main insights of our results extend to a setting in which some links can be formed at no cost.

In most of the analysis below, we focus on investigating the relationship between stable networks and efficient networks. Additionally, we investigate the amount of inequality prevailing in equilibria in our model. For this, we will use the Atkinson class of inequality measures (Atkinson, 1970). Specifically, we consider a welfare function $W : \mathbb{R}^{|\mathbf{N}|} \rightarrow \mathbb{R}$ that maps a profile of expected utilities into the set of real numbers such that

$$(9) \quad W(u) = \sum_{i \in \mathbf{N}} f(u_i),$$

²⁶Note that these definitions are not mutually exclusive (there can be both underinvestment and overinvestment inefficiency) or collectively exhaustive (inefficient networks can have neither underinvestment nor overinvestment inefficiency if an increase in the net total surplus is possible only by the simultaneous addition and removal of links).

where $f(\cdot)$ is assumed to be an increasing, strictly concave, and differentiable function. The concavity of $f(\cdot)$ captures the social planner's preference for more-equal income distributions. Supposing that all agents instead received the same expected utility u' , we pose the question of what aggregate expected utility is required to keep the level of the welfare function constant.²⁷ In other words, we find the scalar $u' : |\mathbf{N}|f(u') = \sum_{i \in \mathbf{N}} f(u_i)$. Letting $\bar{u} = (1/|\mathbf{N}|) \sum_{i \in \mathbf{N}} u_i$ be the mean expected utility, Atkinson's inequality measure (or index) is given by

$$(10) \quad I(f) = 1 - \frac{u'}{\bar{u}} \in [0, 1].$$

We let \mathcal{I} be the set (class) of Atkinson inequality measures and note that any $I(f) \in \mathcal{I}$ equals zero if and only if all agents receive the same expected utility.²⁸ There is an infinite set of inequality measures in the Atkinson class, and two different inequality measures in the class can rank the inequality of two distributions differently. However, there are certain pairs of distributions that are ranked the same way by all members of the class, such as when one distribution is a mean-preserving spread of the other one.

4. WITHIN-GROUP NETWORKS

In this section we assume that $|\mathbf{M}| = 1$, that is, that agents are ex-ante symmetric and that any differences in their outcomes stem from their stable positions on the social network. This will lay the foundations for the more general case considered in the next section.

We begin our investigation by proving a general characterization of the set of stable networks. Recall that a *path* between i and j is a walk in which no agent is visited more than once. If there are K paths between i and j on network L , we let $\mathbf{P}(i, j, L) = \{P_1(i, j, L), \dots, P_K(i, j, L)\}$ be the set of these paths. For every $k \in \{1, \dots, K\}$, let $|P_k(i, j, L)|$ be the cardinality of the set of agents on path $P_k(i, j, L)$.²⁹ We can now use these definitions to define a quantity that captures how far apart two agents are on a network in terms of the probability that for a random arrival order they will be connected without a direct link when the second of the two agents arrives. We will refer to this distance as the agents' Myerson distance:

$$(11) \quad md(i, j, L) := \frac{1}{2} - \sum_{k=1}^{|\mathbf{P}(i, j, L)|} (-1)^{k+1} \left(\sum_{1 \leq i_1 < \dots < i_k \leq |\mathbf{P}(i, j, L)|} \left(\frac{1}{|P_{i_1} \cup \dots \cup P_{i_k}|} \right) \right).$$

²⁷This exercise is analogous to the certainty-equivalent exercise that can be undertaken for an agent facing stochastic consumption.

²⁸As $f(\cdot)$ approaches a linear function, the social planner cares less about inequality and $I(f) \rightarrow 0$. Nevertheless, strict concavity prevents $I(f)$ equaling 0 unless all agents receive the same expected utility.

²⁹For example, for a path $P_k(i, j, L) = \{i, i', i'', j\}$, $|P_k(i, j, L)| = 4$, and for a path $P_{k'}(i, j, L) = \{i, i', i''', i'''', j\}$, $|P_{k'}(i, j, L)| = 5$. Finally, we will let $|P_k(i, j, L) \cup P_{k'}(i, j, L)|$ denote the number of different agents on paths $P_k(i, j, L)$ and $P_{k'}(i, j, L)$ combined. (In the example, there are six different agents: i, j, i', i'', i''' , and i'''' .)

This expression calculates the probability that for a random arrival order the link l_{ij} will be essential immediately after i arrives,³⁰ using the classic *inclusion–exclusion principle* from combinatorics. This probability is important because it affects i ’s incentives to link to j .

The Myerson value allocates to each agent her average marginal contribution to total surplus, where the average is taken over all possible arrival orders. For example, for the network shown in Figure 2a consider the arrival order 1, 2, 5, 6, 3, 4. When agent 1 is added, there are no other agents and so no links are formed. Thus 1’s marginal contribution to total surplus is 0. Then agent 2 is added, and the link l_{12} is formed. This link is essential on this network, permitting risk sharing between agents 1 and 2 that wasn’t previously possible. As a result, by equation (4), the total surplus generated by risk sharing increases from 0 to V . Thus 2’s marginal contribution to total surplus for this arrival order is V . When 5 and 6 are added, no new links are formed and no additional risk sharing is possible—their marginal contributions are 0. However, the arrival of 3 results in the formation of the links l_{23} , l_{35} , and l_{36} . All of these links are essential, and risk sharing among agents 1, 2, 5, 6, and 3 becomes possible. This increases total surplus to $4V$ by equation (4), so 3’s marginal contribution to total surplus is $3V$. Finally, when 4 is added the links l_{14} and l_{45} are formed, and this permits risk sharing to also include 4, increasing total surplus to $5V$. Thus 4’s marginal contribution to total surplus is V .

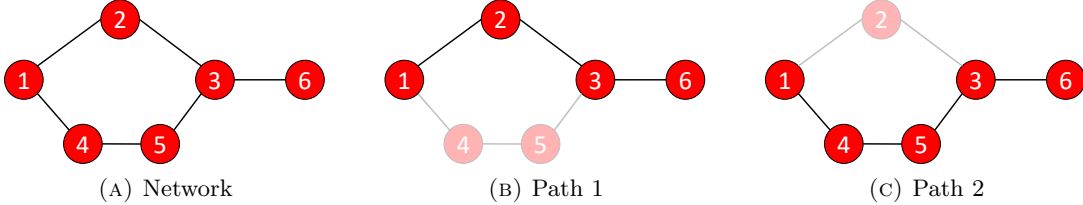


FIGURE 2. Paths connecting nodes 1 and 6

Whenever a link is formed that is essential for a given arrival order, it contributes V to total surplus, while whenever a link is superfluous for a given arrival order, it makes a marginal contribution of 0 to total surplus.³¹ Consider now the incentives agent 1 has to form a superfluous link to agent 6. To calculate this, we need to know the probability with which such a link would be essential for a random arrival order. There are three ways in which the link l_{16} might not be essential upon 1’s arrival. First, with probability $1/2$ agent 6 arrives after agent 1 and the link l_{16} will be formed on 6’s arrival instead of 1’s. Second, Path 1 shown in Figure 2b might be present. This will be the case if and only if agents 2, 3, and 6 arrive before agent 1. The probability that agent 1 is the last of these four agents to arrive is $1/4$. Finally, Path 2 shown in Figure 2c might be present. This occurs if and only

³⁰If, for a given arrival order, the agents in $\mathbf{S} \subseteq \mathbf{N}$ arrive before i , then l_{ij} is essential immediately after i arrives if it is essential on the network $L(\mathbf{S} \cup \{i\})$.

³¹Note that in the arrival order considered in the preceding paragraph, 4’s marginal contribution to total surplus would still have been V without the link l_{14} (l_{45}) as long as the link l_{45} (l_{14}) was still formed.

if agents 3, 4, 5, and 6 arrive before 1. The probability that 1 is the last of these five agents to arrive is $1/5$.

If these three possibilities were mutually exclusive, then the probability that the link l_{16} would be formed and essential upon 1's arrival would be $1 - (1/2) - (1/4) - (1/5)$. The probability that agent 6 arrives after agent 1 is mutually exclusive from the probability that either Path 1 or Path 2 is present, because both these paths need agent 6 to arrive before agent 1. However, it is possible for both Path 1 and Path 2 to be formed upon 1's arrival. Indeed, this occurs if and only if agent 1 is the last agent to arrive, which happens with probability $1/6$. Thus the probability that at least one of the two paths to agent 6 is present upon 1's arrival is $(1/4) + (1/5) - (1/6)$. We need to subtract the probability $1/6$ to avoid double counting the event that both paths are present. Thus the probability that the link l_{16} will be essential upon 1's arrival, is $1 - (1/2) - (1/4) - (1/5) + (1/6) = md(1, 6, L)$.

Lemma 4. *If all agents are from the same group, network L is pairwise stable if and only if*

$$\begin{aligned} (i) \quad md(i, j, L \setminus \{l_{ij}\}) &\geq \kappa_w/V && \text{for all } l_{ij} \in L, \text{ and} \\ (ii) \quad md(i, j, L) &\leq \kappa_w/V && \text{for all } l_{ij} \notin L. \end{aligned}$$

The proof is relegated to Appendix I. Recall from equation (3) that the social benefit of a link is proportional to the variance reduction it generates. For a single group, if a link l_{ij} is essential in the network $L \cup \{l_{ij}\}$, then this variance reduction is $\Delta \text{Var}(L \cup \{l_{ij}\}, L) = (1 - \rho_w)\sigma^2$.

The crucial feature of this expression is that it does not depend on the sizes of the network components that the link l_{ij} connects on L . Although in general the sizes of these components do affect the consumption variance, the two effects exactly offset each other.³² On the one hand, in larger components there are more people to benefit from the essential link. On the other hand, people are already able to smooth their consumption more effectively.

As the social value of a non-essential (superfluous) link, is always zero, the total surplus generated by a network L takes a very simple form. Let $\Upsilon(L)$ be the number of network components on L . Then

$$(12) \quad TS(L) = CE(\Delta \text{Var}(L, \emptyset)) = (|\mathbf{N}| - \Upsilon(L)) \frac{\lambda}{2} (1 - \rho_w) \sigma^2 = (|\mathbf{N}| - \Upsilon(L)) V.$$

Since the surplus created by any essential link is V , the total gross surplus is equal to the product of this constant and the number of network component reductions obtained relative to the empty network.

To consider individual incentives to form links, we can use the definition of the Myerson value and consider the average marginal contribution an agent makes to the total surplus

³²Let $L(\mathbf{S}_1)$ and $L(\mathbf{S}_2)$ be the network components of agent i and agent j , respectively, on network $L \setminus \{l_{ij}\}$, and let $|\mathbf{S}_1| = s_1$ and $|\mathbf{S}_2| = s_2$. Then the sums of the consumption variances on $L(\mathbf{S}_1)$ and $L(\mathbf{S}_2)$ (with Pareto efficient risk sharing) are $\frac{s_1 + s_1(s_1 - 1)\rho_w}{s_1} \sigma^2$ and $\frac{s_2 + s_2(s_2 - 1)\rho_w}{s_2} \sigma^2$, respectively. Once S_1 and S_2 are connected through l_{ij} , the sum of the consumption variances on $L(\mathbf{S}_1 \cup \mathbf{S}_2)$ becomes $\frac{s_1 + s_2 + (s_1 + s_2)(s_1 + s_2 - 1)\rho_w}{s_1 + s_2} \sigma^2$. This implies that the consumption variance reduction induced by the link l_{ij} is $\Delta \text{Var}(L \cup \{l_{ij}\}, L) = (1 - \rho_w)\sigma^2$.

over all possible arrival orders. Specifically, we want to consider the increase in i 's Myerson value due to a link l_{ij} . The link l_{ij} will reduce the number of components in the network by one when i arrives, relative to the counterfactual component reduction without l_{ij} , if and only if j has already arrived and there is no other path between i and j . In other words, the link increases i 's marginal contribution to total surplus if and only if it is essential when i is added. Moreover, for the permutations in which l_{ij} is essential, it contributes V to i 's marginal contribution to total surplus. Averaging over arrival orders, the value to i of the link $l_{ij} \in L$ is $md(i, j, L \setminus \{l_{ij}\})V$, while the value to establishing a new link $l_{ij} \notin L$ is $md(i, j, L)V$.

If a link l_{ij} is essential on L , then for any arrival order there will always be a component reduction of 1 when the later of i or j is added. Therefore, $md(i, j, L) = 1/2$, and l_{ij} will be formed as long as $V > 2\kappa_w$. As V is the social value of forming the link and $2\kappa_w$ is the total cost of forming it, when all agents are from the same group there is never underinvestment in a stable network or overinvestment in an essential link.

Proposition 5. *If all agents are from the same group, then there is never underinvestment in a stable network. Furthermore, there is never overinvestment in an essential link.*

The proof is relegated to Appendix I. When all agents are from the same group, Proposition 5 establishes that there is never overinvestment in an essential link, but overinvestment in superfluous links is possible. If the costs of link formation are low enough, then agents will receive sufficient benefits from establishing superfluous links to be incentivized to do so. Even if a link l_{ij} is superfluous on L , for some arrival orders it will be essential on the induced subnetwork at the moment when i is added and will make a positive marginal contribution to total surplus.³³ An example of such overinvestment is shown in Section E of the Supplementary Appendix.

An immediate implication of Proposition 5 is that if all agents are from the same group and $2\kappa_w > V$, then the only stable network is the empty one and this network is efficient, while if $2\kappa_w < V$ then all stable networks have only one network component (all agents are path connected). For the remainder of the paper, we focus on the parameter range for which the empty network is inefficient for a single group and assume that $2\kappa_w < V$. We refer to this as our *regularity condition* and omit it from the statement of subsequent results.

Under this regularity condition, the set of efficient networks is the set of tree networks in which all agents are path connected. In other words, all agents must be in the same component and all links must be essential. We will now focus on which, if any, of these efficient networks are stable. As agents are well incentivized to form essential links, the only reason an efficient network will not be stable is if two agents have a profitable deviation by forming an additional (superfluous) link.

Figure 3 illustrates three networks: a line (Figure 3a); a circle (Figure 3b) and a star (Figure 3c). While the line and star networks are efficient, the circle network is not, as it

³³Consider, for example, arrival orders in which j arrives first and i arrives second.

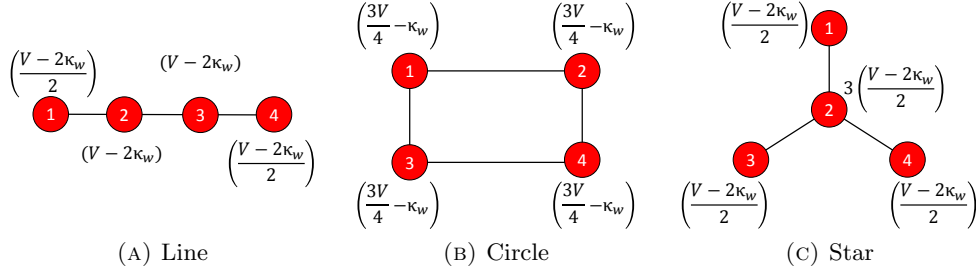


FIGURE 3. Three possible network structures for connecting four agents and the resulting net payoffs

includes a superfluous link. Of the two efficient networks, the star is more stable than the line. Applying Lemma 4, whenever the line is stable so is the star, but there are parameter values for which the star is stable and the line is not. While the star is more stable than the line, it also results in more inequality. The expected utility distribution obtained on the line network can be generated from that obtained on the star network by the best-off agent (agent 2) transferring $(V - 2\kappa_w)/2 > 0$ units of expected utility to one of the worst-off agents (e.g., agent 3). This is enough to ensure that the expected utility distribution on the star is more unequal than the expected utility distribution on the line for any inequality measure in the Atkinson class. We generalize these insights in Proposition 6.

Proposition 6. *Suppose all agents are from the same group.*

- (i) *If there exists an efficient stable network, then star networks are stable, and for a non-empty range of parameter specifications the only networks that are stable are star networks. If a line network is stable, then all efficient networks are stable.*
- (ii) *For all inequality measures in the Atkinson class, among the set of efficient networks, star networks and only star networks maximize inequality, while line networks and only line networks minimize inequality.*

The proof is in Appendix I, but we provide some intuition after we discuss the result. Proposition 6 states that, in a certain sense, among the set of efficient networks the star is the most stable but maximizes inequality, while the line minimizes inequality but is least stable. This indicates a novel tension between stability/efficiency and inequality. For example, in contrast, Pycia (2012) studies when stable coalitional structures exist and finds that stable coalitions are more likely to exist when the bargaining functions of agents are more equal.

To gain intuition for Proposition 6, recall that an efficient network will be stable if and only if no pair of players have a profitable deviation in which they form a superfluous link. By Lemma 4, the incentives for two agents to form such a link are strictly increasing in their Myerson distance. Thus a network is stable if and only if the pair of agents furthest apart from each other, in terms of their Myerson distance, cannot benefit from forming a link. As efficient networks are tree networks, the Myerson distance between any two agents depends

only the length of the unique path between them.³⁴ The longest path between any pair of agents is, by definition, the diameter of the network, $d(L)$. Thus, an efficient network is stable if and only if its diameter is sufficiently small. More precisely, an efficient network L is stable if and only if its diameter is weakly less than $\bar{d}(\kappa_w, V)$, where $\bar{d}(\kappa_w, V)$ is increasing in κ_w , decreasing in V , and integer valued.³⁵

Let $\mathcal{L}^e(\mathbf{N})$ be the set of efficient networks. Star networks have the smallest diameter among networks within this set, while line networks have the largest diameter among networks within this set. This establishes part (i) of Proposition 6.

To gain intuition for part (ii), a first step is noting that on any efficient network agents' net payoffs are proportional to their degrees (i.e., the number of neighbors they have): $u_i(L) = |\mathbf{N}(i; L)|(V/2 - \kappa_w)$. The key insight is then showing that for any network in the set of efficient networks $\mathcal{L}^e(\mathbf{N})$, the star network can be obtained by rewiring the network (deleting a link $l_{ij} \in L$ and adding a link $l_{ik} \notin L$) in such a way that at each step we increase the degree of the agent who already has the highest degree, reduce the degree of some other agent, and obtain a new network in $\mathcal{L}^e(\mathbf{N})$. This process transfers expected utility to the agent with the highest expected payoff from some other agent, thereby increasing inequality for any inequality measure in the Atkinson class. Likewise, we can obtain the line network from any network in the set $\mathcal{L}^e(\mathbf{N})$ by rewiring the network to decrease the degree of the agent with the highest degree at every step. This transfers expected utility from the agent with the highest expected payoff to some other agent, thereby decreasing inequality for any inequality measure in the Atkinson class.

To summarize, this section identifies a novel downside to informal risk-sharing agreements. Even when investments into social capital are efficient, the networks that can be supported in equilibrium generate social inequality, and this translates into (potentially severe) financial inequality. The setting we have used to identify this tension between efficiency/stability and equality is very stylized in a number of dimensions. In the rest of the paper, we extend our baseline model in a variety of directions to demonstrate that this basic tension is extremely robust. First, we partition the set of agents into multiple groups and generalize the joint income distribution (Section 5). Then in Section 6 we show robustness to (i) simultaneously generalizing the key assumptions in our baseline model while retaining its basic structure; and (ii) incorporating enforcement. We also show in this section that the same tension appears in an alternative model with imperfect risk sharing and demonstrate a fundamental trade-off between equality and efficiency/stability that is present extremely generally by building on some well-known graph-theory results.

³⁴Suppose d is the number of agents on the unique path connecting i and j . The probability that this path exists when agent i arrives is $1/d$. In addition, if agent j has not yet arrived, which occurs with probability $1/2$, i would not benefit from the link l_{ij} , so i 's expected payoff from forming a superfluous link to j is $(1 - (1/2) - (1/d))V$. We also note that as d gets large, this converges to $V/2$, which is the value i receives from forming an essential link.

³⁵We show in the proof of Proposition 6 that $\bar{d}(\kappa_w, V) = \lfloor 2V/(V - 2\kappa_w) \rfloor$.

5. CONNECTIONS ACROSS GROUPS

We now generalize our model by permitting multiple groups. These different groups might correspond to people from different villages, different occupations, or different social status groups, such as castes. We will first show that (under our regularity condition) there is still never any within-group underinvestment. However, this does not apply to links that bridge groups. As, by assumption, incomes are more correlated within a group than across groups, there can be significant benefits from establishing such links, and not all these benefits accrue to the agents forming the link. Intuitively, an agent establishing a bridging link to another group provides other members of her group with access to a less correlated income stream, which benefits them. As agents providing such bridging links are unable to appropriate all the benefits these links generate, and these links are relatively costly to establish, there can be underinvestment.

To analyze the incentives to form links within a group, we first need to consider the variance reduction obtained by a within-group link. Such a link may now connect two otherwise separate components consisting of arbitrary distributions of agents from different groups. Suppose the agents in $\mathbf{S}_0 \cup \dots \cup \mathbf{S}_k$ and the agents in $\widehat{\mathbf{S}}_0 \cup \dots \cup \widehat{\mathbf{S}}_k$ form two distinct network components, where for every $i \in \{0, \dots, k\}$ the agents in \mathbf{S}_i and those in $\widehat{\mathbf{S}}_i$ are all from group i . Consider now a potential link l_{ij} connecting the two otherwise disconnected components. Letting s_0 be the number of agents in group 0, the variance reduction obtained is³⁶

$$(13) \quad \Delta \text{Var}(L \cup \{l_{ij}\}, L) = \left[(1 - \rho_w) + \frac{\sum_{i=0}^k \left(\hat{s}_i \sum_{j=0}^k s_j - s_i \sum_{j=0}^k \hat{s}_j \right)^2}{\left(\sum_{i=0}^k s_i \right) \left(\sum_{i=0}^k \hat{s}_i \right) \left(\sum_{i=0}^k s_i + \hat{s}_i \right)} (\rho_w - \rho_a) \right] \sigma^2.$$

The key feature of this variance reduction is that it is always weakly greater than $(1 - \rho_w)\sigma^2$, which is the variance reduction we found in the previous section when all agents were from the same group. Thus the presence of across-group links increases the incentives for within-group links to be formed. A within-group link can now give (indirect) access to less correlated incomes from other groups and so is weakly more valuable. This implies that there will still be no underinvestment under our regularity condition that $2\kappa_w < V$ is satisfied.³⁷ The above reasoning is formalized by Proposition 7.

Proposition 7. *There is no underinvestment between any two agents from the same group in any stable network.*

³⁶By definition,

$$\Delta \text{Var}(L \cup \{l_{ij}\}, L) = \text{Var}(L(\mathbf{S}_0, \dots, \mathbf{S}_k)) + \text{Var}(L(\widehat{\mathbf{S}}_0, \dots, \widehat{\mathbf{S}}_k)) - \text{Var}(L(\mathbf{S}_0 \cup \widehat{\mathbf{S}}_0, \dots, \mathbf{S}_k \cup \widehat{\mathbf{S}}_k)).$$

Recalling that

$$\text{Var}(L(\mathbf{S}_0, \mathbf{S}_1, \dots, \mathbf{S}_k)) = \left(\sum_{i=0}^k (s_i + s_i(s_i - 1)\rho_w) + 2\rho_a \sum_{i=0}^{k-1} \left(s_i \sum_{j=i+1}^k s_j \right) \right) \sigma^2 / \sum_{i=0}^k s_i,$$

some algebra yields the result.

³⁷Recall that this regularity condition requires only that it be efficient for two agents in the same group, both without any other connections, to form a link.

The proof of Proposition 7 is in Appendix I. While underinvestment is not possible within a group, it is possible across groups. An example of this is shown in Section E of the Supplementary Appendix. Although when all agents are from the same group the value of an essential link does not depend on the sizes of the components it connects, the value of an essential link connecting two different groups of agents increases in the sizes of the components. To demonstrate this formally, consider an isolated group that has no across-group connections, and consider the incentives for a first such connection to be formed. Thus the first component consists of agents from a single group, say group 0. We let the second component consist of agents from one or more of the other groups (1 to k). The variance reduction obtained by connecting these two components is

$$(14) \quad \Delta \text{Var}(L \cup \{l_{ij}\}, L) = \left[(1 - \rho_w) + \frac{\hat{s}_0 \left(\left(\sum_{i=1}^k s_i \right)^2 + \sum_{i=1}^k s_i^2 \right)}{\left(\sum_{i=1}^k s_i \right) \left(\hat{s}_0 + \sum_{i=1}^k s_i \right)} (\rho_w - \rho_a) \right] \sigma^2,$$

which is increasing in \hat{s}_0 :

$$(15) \quad \frac{\partial \Delta \text{Var}(L \cup \{l_{ij}\}, L)}{\partial \hat{s}_0} = \frac{\left(\sum_{i=1}^k s_i \right)^2 + \sum_{i=1}^k s_i^2}{\left(\hat{s}_0 + \sum_{i=1}^k s_i \right)^2} (\rho_w - \rho_a) \sigma^2 > 0.$$

The inequality follows since $\rho_w > \rho_a$. Thus if agents i, j for link l_{ij} that connects two otherwise unconnected groups, they receive a strictly smaller combined private benefit than the social value of the link. To see why, suppose that on network L the link l_{ij} is essential, and without l_{ij} there would be two components, the first connecting agents from group $G(i)$ and the second connecting agents from group $G(j) \neq G(i)$. Consider the Myerson value calculation. For arrival orders in which i or j is last to arrive, the value of the additional variance reduction due to l_{ij} obtained upon the arrival of the later of i or j is the same as its marginal social value, that is, the value of variance reduction obtained by l_{ij} on L . For any other arrival order, the value of variance reduction due to l_{ij} when the later of i or j arrives is strictly less. Averaging over these arrival orders, the link l_{ij} contributes less to i and j 's combined Myerson values than its social value, leading to the possibility of underinvestment.

Besides underinvestment, overinvestment is also possible across groups. Forming superfluous links will increase an agent's share of surplus without improving overall risk sharing and can therefore create incentives to overinvest. Nevertheless, when κ_a is relatively high, underinvestment rather than overinvestment in across-group links will be the main efficiency concern. In many settings, within-group links are relatively cheap to establish in comparison to across-group links. For example, when the different groups correspond to different castes, it can be quite costly to be seen interacting with members of the other caste (e.g., Srinivas (1962), Banerjee et al. (2013b)). Motivated by this, and because across-group links are typically far sparser than within-group links, we focus our attention on this parameter region.

More concretely, below we investigate what within-group network structures create the best incentives to form across-group links and what network structures minimize the incentives for within-group overinvestment. Remarkably, we find that these two forces push within-group network structures in the same direction, and in both cases toward inequality in the society.

We begin by considering within-group overinvestment, which corresponds to the formation of superfluous links within a group. We found in the previous section that when all agents are from the same group the star is the efficient network that minimizes the incentives for overinvestment. However, once we include links to other groups, the analysis is more complicated. The variance reduction a within-group link generates is still 0 if the link is superfluous, but when the link is essential it depends on the distribution of agents across the different groups the link grants access to. Moreover, the variance reduction may be decreasing or increasing in the numbers of people in those groups.³⁸ This makes the Myerson value calculation substantially more complicated. When all agents were from the same group, all that mattered was whether the link was essential when added. Now, for each arrival order in which the link is essential we also need to keep track of the distribution of agents across the different groups that are being connected. Nevertheless, our earlier result generalizes to this setting, although the argument establishing the result is more subtle.

To state the result, it is helpful to define a new network structure. A *center-connected star network* is a network in which all within-group network structures are stars and all across-group links are held by the center agents in these stars. We denote the set of center-connected star networks by \mathcal{L}^{CCS} .

Proposition 8. *If any efficient network L is robust to within-group overinvestment, then any center-connected star network $L' \in \mathcal{L}^{CCS}$ is also robust to within-group overinvestment. Moreover, if $L \notin \mathcal{L}^{CCS}$, then for a range of parameter specifications any center-connected star network $L' \in \mathcal{L}^{CCS}$ is robust to within-group overinvestment but L is not.*

The proof of Proposition 8 is in Appendix I. In Proposition 6 we found that when all agents are from the same group, incentives for (within-group) overinvestment are minimized by forming a (within-group) star. However, the incentives to form superfluous within-group links are weakly greater when someone within the group holds an across-group link (see equation (13)). We can therefore think of the incentives for overinvestment that we found in Proposition 6 as a lower bound on the minimal incentives we can hope to obtain once there are across-group links. A key step in the proof of Proposition 8 shows that this lower bound is obtained by all center-connected star networks.

Consider a center-connected star network L' . As the agent at the center of a within-group star, agent k , has a link to all agents within the same group, we can focus on the incentives of two non-center agents from the same group, i and j , to form a superfluous link. Consider

³⁸In the case of an essential across-group link that connects agents from just one group to agents from other groups, the comparative statics are unambiguous. In this case, the variance reduction is increasing in the sizes of the groups connected (see inequality (15)).

any subset of agents $\mathbf{S} \subseteq \mathbf{N}$ such that $i, j \in \mathbf{S}$. On the induced subnetwork $L'(\mathbf{S})$, either l_{ij} is superfluous or $k \notin \mathbf{S}$. This implies that no across-group links are present whenever the additional link l_{ij} makes a positive marginal contribution. Thus considering different arrival orders, the average marginal contribution of such a link when it is added is the same on the star network with no across-group links as on a center-connected star network: The lower bound on within-group overinvestment incentives is obtained.

We now consider the within-group network structures that maximize the incentives for an across-group link to be formed. We have already established that the marginal contribution of a first bridging link to the total surplus is increasing in the sizes of the groups it connects. By the Myerson value calculation, the agents with the strongest incentives to form such links are then those who will be linked to the greatest number of other agents within their group when they arrive. The result below formalizes this intuition.

Let $\mathcal{A}(\mathbf{S}_k)$ be the set of possible arrival orders for the agents in \mathbf{S}_k . For any arrival order $A \in \mathcal{A}(\mathbf{S})$, let $\mathbf{T}_i(A)$ be the set of agents to whom i is path connected on $L(\mathbf{S}')$, where \mathbf{S}' is the set of agents (including i) that arrive weakly before i . Let $T_i^{(m)}$ be a random variable, taking values equal to the cardinality of $\mathbf{T}_i(A)$, where A is selected uniformly at random from those arrival orders in which i is the m th agent to arrive.

We will say that agent $i \in \mathbf{S}_k$ is more *Myerson central* (from now on, simply more central, for brevity) within her group than agent $j \in \mathbf{S}_k$ if $T_i^{(m)}$ first-order stochastically dominates $T_j^{(m)}$ for all $m \in \{1, 2, \dots, |\mathbf{S}_k|\}$.³⁹ In other words, considering all the arrival orders in which i is the m th agent to arrive and all the arrival orders in which j is the m th agent to arrive, the size of i 's component at i 's arrival is larger than that of j 's at j 's arrival in the sense of first-order stochastic dominance.⁴⁰ This measure of centrality provides a partial ordering of agents.

Lemma 9. *Suppose agents in \mathbf{S}_0 form a network component, and all other agents in \mathbf{N} form another network component. Let $i, i' \in \mathbf{S}_0$, and let $j \notin \mathbf{S}_0$. If i is more central within her group than i' , then i receives a higher payoff from forming l_{ij} than i' receives from forming $l_{i'j}$:*

$$MV(i; L \cup \{l_{ij}\}) - MV(i; L) > MV(i'; L \cup \{l_{i'j}\}) - MV(i'; L).$$

The proof is relegated to Appendix I. The key step in the proof pairs the arrival orders of a more-central agent with a less-central agent, so that in each case the more-central agent is connected to weakly more people in the same group upon her arrival, and to the same set of people from other groups. Such a pairing of arrival orders is possible from the definition of centrality, and in particular the first-order stochastic dominance it requires.

³⁹We also use this notion of centrality to compare the within-group centrality of the same agent on two different network structures. To avoid repetition, we do not state the slightly different definition that would apply to this situation.

⁴⁰An alternative and equivalent definition is that i is more central than j if there exists a bijection $B : \mathcal{A}(\mathbf{S}_k) \rightarrow \mathcal{A}(\mathbf{S}_k)$ such that $|\mathbf{T}_i(A)| \geq |\mathbf{T}_j(B(A))|$ and $A(i) = A'(j)$, where $A(i)$ is i 's position in the arrival order A and $A' = B(A)$.

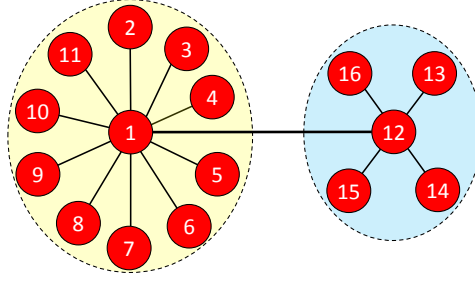


FIGURE 4. Center-connected within-group stars, in a context with two groups.

Lemma 9 shows that more-central agents have better incentives to form across-group links. We can then consider the problem of maximizing the incentives to form across-group links by choosing the within-group network structures (networks containing only within-group links). We will say that the within-group network structures that achieve these maximum possible incentives are most robust to underinvestment inefficiency across groups.

Proposition 10. *If any efficient network L is robust to across-group underinvestment, then some center-connected star network $L' \in \mathcal{L}^{CCS}$ is also robust to across-group underinvestment. Moreover, if $L \notin \mathcal{L}^{CCS}$, then for a range of parameter specifications the center-connected star network $L' \in \mathcal{L}^{CCS}$ is robust to across-group underinvestment but L is not.*

The proof of Proposition 10 is in Appendix I. Intuition can be gained from Lemma 9, which shows that agents have better incentives to provide a bridging link across groups when they are more central within their own group. Thus to maximize the incentives of an agent to provide an across-group link, we need to maximize the centrality of this agent within her group. This is achieved by any network that directly connects this agent to all others in the same group. However, only one of these within-group network structures can be part of an efficient network, and this is the star network, with the agent providing the across-group link at the center.

Figure 4 shows a center-connected star network when there are two groups. As long as it is efficient for these groups to be connected, center-connected star networks and only center-connected star networks minimize the incentives for within-group overinvestment (by Proposition 8) and minimize the incentives for across-group underinvestment (by Proposition 10).

The above results further reinforce the tension between efficiency and equality. However, one subtlety relative to the one-group case is that while the center-connected star network maximizes social inequality, in terms of agents' degrees, among all efficient networks, additional assumptions are required on the parameters of the model to ensure that such networks also maximize income inequality for all inequality measures in the Atkinson class. Taking the link-formation costs as sunk and considering only second-period income, income inequality is maximized by center-connected star networks (among efficient networks). However, while the

formation of the across-group link benefits the center agents and increases their expected net payoffs (including link-formation costs), unlike the formation of other links it also increases the expected net payoffs of other individuals in the group.

6. EXTENSIONS

In the model we presented in the previous sections, we abstracted from many potentially relevant aspects of informal risk sharing, such as enforcement issues or coalitional (group) deviations. Additionally, that model imposes specific functional forms on the income distribution and utility functions, and it assumes a micro-founded but specific rule of sharing the surplus generated by risk sharing. These assumptions and modeling choices were made for analytical tractability. In this section we demonstrate that our model can be generalized and extended in many directions, and yet the main qualitative finding, that there is a tension between equality and efficiency when incentive compatibility is required, remains intact. Our other results, on underinvestment and overinvestment inefficiencies, can also be extended to more general environments than the baseline model, and we provide some results in this direction in the Supplementary Appendix. We limit attention in this section to homogeneous groups. Extensions of results involving multiple groups are addressed in the Supplementary Appendix.

6.1. More general environments and surplus-sharing rules. First, we examine how the main insights from the baseline model extend when we allow for more general income distributions, utility functions, and risk-sharing arrangements.

Outside the CARA-normal specification of the model, expected utilities are in general non-transferable, so we need to take a more general approach to modeling the risk-sharing arrangements. Let $v_i(c_i)$ be the utility function for agent i , mapping second-period consumption into utility. We assume that $v_i = v_j = v$ for all i and j , and that v is strictly increasing and strictly concave. Let \mathcal{P} be the distribution the incomes are drawn from.

Let \mathcal{L} be the set of all possible networks for agents in \mathbf{N} . We assume there is a unique risk-sharing arrangement that will be implemented for any possible network $L \in \mathcal{L}$, and that agents correctly anticipate the risk-sharing agreement that will obtain. These risk-sharing arrangements, which depend on the social network, might be dictated by social conventions, or they can be outcomes of negotiation processes for transfer arrangements once the network is formed. Let $\tau(L)$ be the transfer arrangement, and let $u_i^\tau(L)$ be the expected second-period consumption utility of agent i implied by $\tau(L)$.⁴¹

We continue to assume that for every $L \in \mathcal{L}$, $\tau(L)$ specifies a pairwise-efficient risk-sharing arrangement $\tau_{ij}(L)$ for every pair of agents i, j that are linked in L . As shown earlier, this is equivalent to $\tau(L)$ being Pareto efficient at the component level. Agent i maximizes the difference between her expected utility from the second-period risk sharing (given by u_i^τ) and

⁴¹More precisely, the utility function v_i , the distribution of income realizations, and the transfer arrangement $\tau(L)$ jointly determine $u_i^\tau(L)$.

her costs of establishing links. Let $C_i(L)$ be the set of agents on the same component as i given L .

Next, we impose a series of assumptions on $\tau(\cdot)$. We do not claim that the above assumptions hold universally when informal risk sharing takes place. Our main objective is to demonstrate that our qualitative results hold for a much broader class of models than the CARA-normal setting with surplus division governed by the Myerson value.

Assumption 11.

- (a) For every $i \in N$, $u_i^\tau(L \cup \{l_{ij}\}) > u_i^\tau(L)$ for every $L \in \mathcal{L}$ and all $j \in \mathbf{N}$ such that $l_{ij} \notin L$.
- (b) For every $k \in N$, $u_k^\tau(L \cup \{l_{ij}\}) \geq u_k^\tau(L)$ for every $L \in \mathcal{L}$ and all $i, j \in \mathbf{N}$ such that $C_i(L) \neq C_j(L)$.
- (c) If $l_{ij} \notin L$, then

$$u_i^\tau(L \cup \{l_{ij}\}) - u_i^\tau(L) = g(d(i, j, L), |C_i(L)|, |C_i(L \cup \{l_{ij}\})|),$$

where the function $g(d(i, j, L), |C_i(L)|, |C_i(L \cup \{l_{ij}\})|)$ is increasing in the distance measure $d(i, j, L)$ and $d : \overline{\mathbf{N}}^2 \times \mathcal{L} \rightarrow \mathbb{R}_{++}$ (where $\overline{\mathbf{N}}^2 = \{(i, j) | i, j \in \mathbf{N}, i \neq j\}$) has the following properties:

- (i) If i and j are in different components on L , then $d(i, j, L) = \bar{d}$, with \bar{d} strictly greater than the maximum possible distance between any two path-connected agents.
- (ii) d depends only on paths (thus ignoring walks with cycles).
- (iii) Let S_{ij} be the set of paths between i and j , and let S_{kl} be the set of paths between k and l . We assume that $d(i, j, L) > d(k, l, L)$ if there exists a matching function⁴² $\mu \in \mathcal{M}(S, S')$ such that each path between i and j is matched to a shorter path between k and l , and that all such paths between k and l are independent (do not pass through any of the same nodes as each other).
- (d) For all networks L ,

$$\kappa_w/2 < \min_{i, j: C_i(L) \neq C_j(L)} u_i^\tau(L \cup \{l_{ij}\}) - u_i^\tau(L).$$

We maintain Assumption 11 for the rest of this subsection. Part (a) requires that establishing a link always strictly increases the connecting agents' expected consumption utilities. Part (b) requires that the formation of an essential link imposes no negative pecuniary externalities on other agents. Part (c) extends the idea that the private benefits two agents receive from establishing a link should be increasing in the distance between them, while permitting these private benefits to also depend on the sizes of the components being connected. The notion of distance used in this assumption is relatively broad. The class of distance measures

⁴²For two sets S and S' , we define $\mathcal{M}(S, S')$ as the set of matching functions $\mu : S \rightarrow S' \cup \{\emptyset\}$ such that for $s \in S$, if $\mu(s) \neq \emptyset$ then $\mu(s) \neq \mu(t)$ for all $t \in S \setminus \{s\}$. Thus every $\mu \in \mathcal{M}(S, S')$ maps each element of S either to a different element of S' or to the empty set.

permitted includes the Myerson distance, which was found to matter in our baseline model, as well as many others. Note that the requirement (iii) on the distance measure provides only a weak partial ordering for the distances between agents. Part (d) requires the cost of forming a link to be small relative to the private benefits of establishing an essential link. For general utility functions and transfer arrangements, there is no guarantee that there is not underinvestment. Part (d) restricts attention to parts of the parameter space in which underinvestment is ruled out.

A network is Pareto efficient if there is a feasible transfer agreement that could be reached on that network such that there is no other pair, consisting of a network and feasible transfer agreement in which all agents are weakly better off and some agents are strictly better off.

Proposition 12. *A network is Pareto efficient if and only if it is a tree that connects all agents.*

Note that for any non-essential link l_{ij} , $|C_i(L)| = |C_i(L \cup \{l_{ij}\})|$. Thus the marginal benefits from i and j forming a superfluous link depend only on the distance between i and j on L and the number of agents in their component. The latter includes all agents for any efficient network, by Proposition 12. Thus for an efficient network L , by Assumption 11 part (c), the marginal benefit i and j receive from forming a superfluous link depends only on the length of the unique path between them, and is strictly increasing in this path length. Thus an efficient network will be stable if and only if the maximum distance between any two agents is sufficiently low. The next corollary formally states this result.

Corollary 13. *An efficient network is stable if and only if its diameter is sufficiently small.*

A network is *least* stable within a class of networks when its stability implies the stability of all other networks in that class. A network is *most* stable within a class of networks, when its instability implies the instability of any other network in that class.

Proposition 14.

- (i) *The most stable efficient network is the star.*
- (ii) *The least stable efficient network is the line.*

In this generalized framework, further assumptions are needed to guarantee that the star is the least equitable tree network, and that the line is the most equitable tree network, for *all* inequality measures within the Atkinson class. The next proposition provides one sufficient condition, requiring that if one efficient network can be obtained from another one by rewiring exactly one link, then only the utilities of those agents who gain or lose a link are affected. This condition is satisfied in our benchmark model.⁴³

⁴³In the benchmark model, on any efficient network L the payoff of agent i is proportional to her degree. As for all pairs of efficient networks L and L' such that $L' = (L \setminus \{l_{ij}\}) \cup \{l_{jk}\}$ the degrees of all agents except i and k are held constant by the rewiring, and all agents except i and j receive the same payoffs on L' as on L .

Proposition 15. *Suppose that for all pairs of efficient networks L and L' such that $L' = (L \setminus \{l_{ij}\}) \cup \{l_{jk}\}$, the transfer arrangements satisfy $\tau_l(L) = \tau_l(L')$ for all $l \neq i, k$. Then for all inequality measures in the Atkinson class, among the set of efficient networks, star networks and only star networks maximize inequality, while line networks and only line networks minimize inequality.*

6.2. Enforcement through supported risk sharing and coalitional deviations. Thus far we have abstracted from enforcement problems. In this section we extend the model to capture the idea that having friends in common can reduce an agent's incentives to renege on an agreement. This might be because the friend in common is able to monitor actions and identify the guilty party in a dispute, or because reneging on the agreement will lead to a damaging reputation loss with the friend in common. While it is beyond the scope of this paper to fully explore these issues and there is a vibrant literature that focuses on network-based enforcement of agreements (see, for example, Jackson et al. (2012), Wolitzky (2012), Ali and Miller (2013, 2016), Ambrus et al. (2014), Nava and Piccione (2014), and Ambrus et al. (2016)), in this section, motivated by this literature, we model the value of friends in common for enforcement by assuming that risk sharing between two agents is possible if and only if those two agents have a friend in common. This is known as closure (Coleman, 1988) and has long been thought important for cooperation because it enables collective sanctions to be imposed on a deviating agent—if an agent cheats on one of their neighbors, there are friends in common that can also punish the deviating agent.

A link in L is supported and can be used for risk sharing if and only if it is part of a triangle (i.e., the complete network among three agents). Let $L'(L)$ be the spanning subgraph of L which contains only supported links. An illustration of this is provided in Figure 5. Risk-sharing arrangements and rent distribution are as in Section 2. The only difference is that now risk sharing takes place on the network $L'(L)$ instead of L (but agents continue to pay to form links in L). As in this setting it takes more than two agents to facilitate risk sharing, we also require robustness of the network to the minimal coalitional deviations necessary to prevent the empty network from being stable.

Before we can state our main result for this section, we need some new terminology. A network L is a *tree union of triangles* if it can be expressed as the union of $m \geq 2$ (non-node-disjoint) subnetworks ordered as $L(N_1), \dots, L(N_m)$, such that $\cup_{i=1}^{m-1} N_i \cap N_{k+1} = 1$ and each subnetwork $L(N_i)$ is a triangle. Thus each subnetwork in the sequence is a triangle that has exactly one node in common with the union of all the nodes in the subnetworks preceding it in the sequence. Figures 6a and 6b illustrate two tree unions of triangles. The tree union of triangles illustrated in Figure 6a is known as a *Friendship graph* or *Windmill network*; in such networks all triangles have the same node in common.

The cost of forming a link is κ , and also as before, we continue to focus on the parameter range for which risk sharing among all agents is always efficient. As before, the surplus obtained from enabling risk sharing among two groups of agents is V . Proposition 16 shows

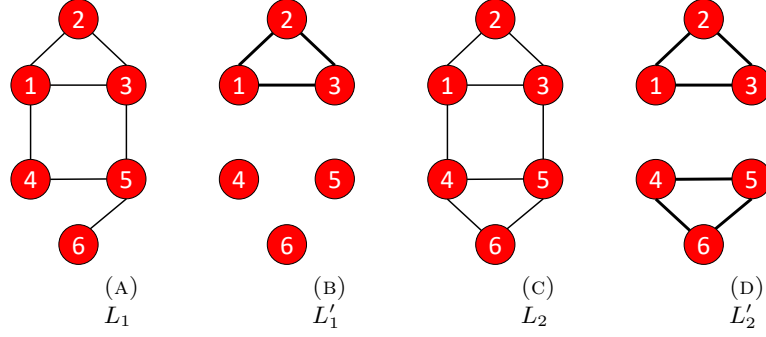


FIGURE 5. (A) Example of a network, L_1 , among six villagers. (B) The links on L_1 that support risk sharing. (C) Example of a different network, L_2 , among six villagers. (D) The links on L_2 that support risk sharing.

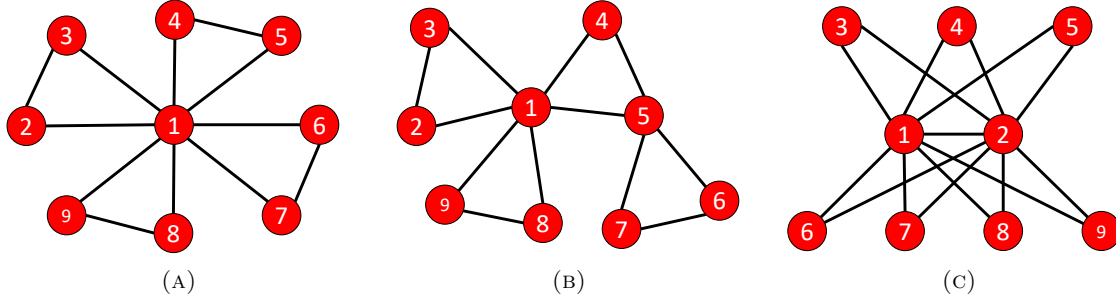


FIGURE 6. (A) The friendship graph connecting nine villagers. (B) A different tree union of triangles connecting nine villagers. (C) An alternative network connecting 9 villagers in which all villagers are able to risk share but all triangles share a common link, l_{12} , hence it isn't a tree union of triangles.

that it is efficient for all agents to risk share if and only if $V \geq 3\kappa$, and that the efficient networks are then tree unions of triangles. Thus in comparison to Section 2, where agreements didn't need to be supported to be enforceable, tree unions of triangles play the role of tree networks.

Proposition 16. *Suppose the number of villagers $n \geq 3$ is odd.*

- (i) *If risk sharing among all n agents is efficient, then the efficient risk-sharing networks are tree unions of triangles.*
- (ii) *Risk sharing among all n agents is efficient for all n if and only if $V \geq 3\kappa$.*

All proofs for this section are relegated to the Supplementary Appendix. To gain intuition for this result, first observe that any link that is not supported is costly to form but cannot be used for risk sharing. While in principle such a link might still be valuable as a means for supporting an agreement on another link, this requires a triangle to be formed with that other link which would make it supported. Thus in an efficient network every links must be

supported and must be part of some triangle. Given this, the most efficient way to organize links (among an odd number of agents) is to form a tree union of triangles. This creates distinct triangles in which no link is shared by two triangles. As a comparison, note that there are 15 links formed in the network depicted in Figure 6c, where all risk-sharing triangles share a common link, while there are only 12 links in the tree unions of triangles illustrated in Figures 6a and 6b.

Jackson et al. (2012) find a class of networks they call social quilts to be those that can supporting risk-sharing agreements based on renegotiation proofness. Interestingly, tree unions of triangles are social quilts. The networks we identify through efficiency considerations based on the very simple condition of support being necessary for risk sharing, would also be renegotiation proof in their setting. This provides further motivation for the simple approach to enforcement we take.

We now consider the stability of the efficient risk-sharing networks. Because of the need for groups of at least three to support risk sharing, here we require stability with respect to a simple form of coalitional deviations for groups of three agents.⁴⁴ In particular, we call a network *tripletwise stable* with respect to expected utilities $\{u_i(L)\}_{i \in \mathbf{N}}$ if and only if it is pairwise stable and for all $i, j, k \in \mathbf{N}$, if two or more of l_{ij} , l_{ik} , l_{kj} are not in L and \hat{L} is the union of network L with these three links, and if $u_i(\hat{L}) \geq u_i(L)$ and $u_j(\hat{L}) \geq u_j(L)$ with at least one of these inequalities strict, then $u_k(\hat{L}) < u_k(L)$. In words, tripletwise stability requires a network to be pairwise stable and does not allow any set of three players to be able to benefit by forming the remaining links among themselves (thereby facilitating direct risk sharing among themselves).

Proposition 17.

- (i) *If there exists an efficient tripletwise stable network then all friendship networks are tripletwise stable, and for a non-empty range of parameter specifications the only efficient networks that are tripletwise stable are friendship networks.*
- (ii) *For all inequality measures in the Atkinson class, among the set of efficient networks, friendship networks maximize inequality and are the only efficient networks that maximize inequality.*

This result is analogous to results in Proposition 6 in Section 4. There, a star network was the most efficient stable network, but also the most unequal. Proposition 17 shows that this result generalizes to the case in which links must be supported to facilitate risk sharing, but with friendship networks (tree networks of triangles, with one agent in the center being part of every triangle) taking the place of star networks.

The basic intuition for this result mirrors the intuition for Proposition 6. Groups of three agents have stronger incentives to deviate and form links among themselves to facilitate risk sharing when they are further apart. Among the set of efficient networks, the relevant

⁴⁴In Section B of the Supplementary Appendix, we also provide results for pairwise stability.

distances are minimized by the friendship network. In terms of inequality, it can be shown that agents' net payoffs are again proportional to their degrees, and that the total number of links is constant for all tree unions of triangles connecting n agents. Further, in any tree union of triangles all agents must have degree at least 2. The friendship network therefore minimizes the possible degree for all but one agent while maximizing the possible degree for the remaining agent. Analogous arguments to those in Section 4 establishing that the star network maximizes inequality among all efficient networks for all inequality measures in the Atkinson class can then be made to show that in the current setting the friendship network maximizes inequality among all efficient networks for the same class of inequality measures.

6.3. Imperfect risk sharing. The benchmark model assumes that risk sharing is perfect on a connected component of the network. In reality, risk sharing can be imperfect for various reasons. In this section we briefly explore two of those reasons: because state-dependent transfers are costly and because relationships sometimes fail to function. We demonstrate that this can actually provide further motives for agents to form highly centralized networks. When state-dependent transfers are costly, short paths are efficient, and that requires central agents. When relationships fail, alternative paths are required, and providing alternative paths efficiently (with relatively few links) again demands central agents.

6.3.1. Costly transfers. One possible reason for imperfectness of risk sharing is that state-dependent bilateral transfers are costly to make (for example, income realizations might need to be verified, and this could be costly). Here we consider a very simple environment with this feature. Assume there are n agents, with quadratic preferences.⁴⁵ Income realizations are such that exactly one randomly selected agent (agent i) gets hit by a bad shock ($e_i = -1$), one randomly selected agent (agent j) receives a good shock ($e_j = 1$), and all other agents are not hit by any shock. We assume that an ex-ante meeting has to take place between neighboring agents, in order to establish risk-sharing arrangements, and that state-independent transfers can be arranged at this point. Up to this point, only link-formation costs have been incurred. Given these state-independent transfers let $c_i(L)$ denote agent i 's baseline consumption level. However, any subsequent state-dependent (post-income-realization) transfer requires an extra meeting or transaction (and possibly state verification) that incurs a cost $k > 0$. If the cost of forming a link, κ , is small enough, and k is small enough relative to κ , then any efficient risk-sharing network has to be a tree (small κ implies that an efficient network has to be connected, and small k relative to κ implies that duplicate links are inefficient). In this environment it is easy to show that the constrained Pareto-efficient risk-sharing arrangement implies that for any income realization there is a single chain of transfers corresponding to the unique path from the agent who received a good shock to the agent hit by a bad shock, and that everyone along this chain ends up with consumption $c_i - \frac{l}{l+1}k$, where l is the length of the path (i.e., agents along the chain equally divide the costs of the l bilateral transfers

⁴⁵This keeps expected utilities transferable given that incomes will no longer be normally distributed.

required to reach the agent with the bad shock). Given such risk-sharing agreements, the tree network that maximizes social efficiency is the one that minimizes the average distance (path length) between two agents. The average distance is proportional to the Wiener index (the sum of distances between different pairs of agents), and it is well known (see for example Dobrynin, Entringer and Gutman (2001), p.213) that the n -node star is the unique network among n -node trees that minimizes the Wiener index. Thus when transfers are costly, there is an extra reason to expect more-centralized networks, because with a limited number of links such networks minimize the average distance between agents, and hence reduce the costs of risk sharing. As opposed to the benchmark model, here the star is not only the most stable efficient network structure, but in fact the only efficient network structure.⁴⁶

6.3.2. Link failures. Another potential reason for imperfect risk-sharing is that links might fail with some probability, making it impossible to send transfers through them, representing a fallout between neighbors or the absence or unavailability of a neighbor for other reasons.⁴⁷ If such link failures are not uncommon, efficiency requires networks that are denser than tree networks, with multiple paths existing between the same two agents, providing alternative routes to connect them in case link failures cut some of the paths between them. Characterizing efficient networks in such an environment analytically is a hard problem. However, Peixoto and Bornholdt (2012), in a related model, using a combination of analytical and numerical techniques, find that for a large enough number of agents efficient networks have a core-periphery structure, where a small core of nodes with high degree is responsible for most of the connectivity, serving as a central backbone to the system. Such networks are close to multi-center generalizations of the star network, in which all the centers are connected to every agent, while peripheral agents are connected only to the centers.⁴⁸

6.4. Permitting some free links. In practice, relationships are formed for many reasons, and there will be some relationships that exist for reasons unrelated to risk sharing but nevertheless permit risk sharing. These links might, for example, represent family relationships or close friendships formed in childhood. In effect, these are relationships that are formed at no cost for the purpose of risk sharing, providing another explanation for why real-world risk-sharing networks are denser than tree networks. We extend our baseline model to permit this possibility.

Let \hat{L} denote the exogenously given set of links that can be formed at no cost. As, by the Myerson value calculation, a link strictly increases the expected utility an agent

⁴⁶Moreover, if villagers who are more important to implementing the risk-sharing transfers (i.e., those that are involved in more of the transfers) extract more of the surplus from risk sharing, the star will again be associated with the most inequality.

⁴⁷For models of network formation in environments outside the risk-sharing framework with the possibility of probabilistically failing links, see, for example, Bala and Goyal (2000) and Haller and Sarangi (2005).

⁴⁸The efficiency of such network structures requires link failures to be exogenous and independent. If there is a strategic adversary selecting which links fail, more-decentralized networks become optimal (see Dobrynin et al (2001) and Haller and Sarangi (2005)). However, this scenario is less relevant in the risk-sharing context.

receives from the risk-sharing arrangement, we assume that all such links are always formed. The network \hat{L} will consist of a set of components. For each component C , we identify an agent $i^*(C) \in \operatorname{argmin}_i \max_j md_{ij}(C)$. This is an agent who has the lowest maximum Myerson distance to any other agent in component C . We will refer to agent $i^*(C)$ as the *Myerson distance central agent* in component C and let C_i denote the component to which i belongs. Considering all components, we then have a set of Myerson distance central agents $I^* = (i^*(C))_C$. Finally, we identify a Myerson distance central agent associated with the largest distance, $i^{**} \in \operatorname{argmax}_{i^* \in I^*} \max_{j \in C_{i^*}} md_{i^*j}$.

We dub a network generated by forming all free links, and the links $l_{i^*i^{**}}$ for all $i^* \in I^* \setminus \{i^{**}\}$, a *central-connections network*. Central-connections networks are always efficient.⁴⁹ They are the most stable networks within the class of efficient networks.

Proposition 18. *Suppose there is one group. If any efficient network is stable, then all central-connections networks are stable.*

Proposition 18 shows that when some links are formed at no cost, the most stable efficient network forms all additional links required for risk sharing with a single agent. As payoffs are proportional to degree, this again pushes villages toward inequitable outcomes.

6.5. General tensions between stability, efficiency, and equality. In this subsection we point out a general fundamental tension between equality and efficient stable networks. We begin by relating different graph-theoretic concepts to stability, efficiency, and equality.

6.5.1. Equality. We would like to say something general about inequality for all inequality measures in the Atkinson class on formed networks for *any* symmetric payoff function $u : L \rightarrow \mathbb{R}$. Unfortunately, without further restrictions on how network positions translate into payoffs, it is impossible to compare two networks in general. However, it is possible to pose and answer in general the question of when payoffs will be guaranteed to be perfectly equitable.

We proceed under the assumption that only agents' network positions matter for their payoffs—specifically, we require agents in identical network positions to receive the same payoffs. Intuitively, then, if all agents are in identical positions, they must receive equal payoffs. The set of networks for which this holds, thereby guaranteeing perfectly equitable outcomes, will be a useful benchmark that helps identify a general tension between equality and efficiency/stability.

In order to formalize the idea that agents are in identical network positions, we need to introduce some graph theory notation and terminology. We limit attention to connected networks. Every network is implicitly labeled, and we identify the set of labels with the set of nodes \mathbf{N} . Two networks L_1 and L_2 are called *isomorphic*, written $L_1 \sim_I L_2$, if they coincide

⁴⁹As before, the same set of risk-sharing arrangements can be implemented on any given component, and as expected utility is transferable, given that formation costs have been minimized, any point on the Pareto frontier can be reached.

up to labeling, that is, up to a permutation of \mathbf{N} . They are also *automorphic* if given the permutation of nodes associated with the isomorphism f , $l_{f(i)f(j)} \in L_2$ if and only if $l_{ij} \in L_1$. When networks L_1 and L_2 are *automorphic* we write $L_1 \sim_A L_2$. A simple undirected binary graph $L \in \mathcal{L}$ is *vertex transitive* if for every given pair of nodes i and j in \mathbf{N} , there exists an automorphism $f : \mathbf{N} \rightarrow \mathbf{N}$ such that $f(i) = j$. Thus when a network is vertex transitive, we can take a node i and map it to the position of any other node j , by changing the label of j to i , and there exists a way of relabeling the other nodes such that all nodes have exactly the same neighbors as before and the structure of the graph is preserved. Thus the positions of any two nodes i and j in a vertex-transitive network are equivalent in a certain sense, and it is intuitive that the agents should receive the same payoff.

Indeed, in Section D of the Supplementary Appendix, we show that allowing for a general payoff function, vertex transitivity is sufficient for all nodes to receive the same payoff, and that generically, it is also necessary.

Vertex transitivity is a strong condition to place on the network structure. All vertex-transitive networks are regular, but not all regular networks are vertex transitive. Thus the symmetry condition required for perfectly equitable outcomes in general is stronger than the symmetry condition needed in our baseline model.

6.5.2. Efficiency. A network $L = (n, L)$ is *Pareto efficient* if there is no network L' such that the payoffs of the agents on the network $L' = (n, L')$ Pareto dominate those on L (i.e., all agents receive weakly higher net payoffs on L' than on L , and at least one agent receives a strictly higher payoff).

To get a handle on the set of *Pareto-efficient* networks, we assume that shorter path lengths facilitate weakly better risk sharing.

Assumption 19. *All Pareto-efficient networks $L = (n, L)$ have one component, and there is no alternative network $L' = (n, L')$ such that $|L'| \leq |L|$ and the path length distribution of L' first-order stochastically dominates the path length distribution of L .*

This enables us to eliminate some configurations as being Pareto efficient. We also make an assumption that risk-sharing relationships are sufficiently costly to maintain that dense risk-sharing networks are Pareto inefficient.

Assumption 20. *There are no Pareto-efficient networks $L = (n, L)$ in which the average degree $|L|/n \geq \sqrt{n-1}$.*

To aid interpretation, a realistic lower bound on the size of a typical village is 100, while a realistic upper bound is 500. Thus when $n = 100$ this rules out Pareto-efficient risk-sharing networks with an average degree of more than about 10 links, while for $n = 500$ it rules out Pareto-efficient risk-sharing arrangements with an average degree of more than about 22 links. As a comparison, using the data collected by Banerjee, Chandrasekhar, Duflo and Jackson (2013) across 75 rural villages in southern India, the mean number of households in

a village is 209 and the average number of risk-sharing relationships a household has is less than 4. Although there is no guarantee that the networks we observe in practice are efficient, or even close to efficient, the fact that risk-sharing networks are much sparser in practice than required by our upper bound is suggestive that the costs of forming links are sufficiently high to make Assumption 20 reasonable. Moreover, if there is no underinvestment in links in stable networks, as in our benchmark model, then the observed density of links is a valid upper bound for the density of links in Pareto-efficient networks.

6.5.3. Stability. Finally, we turn to stability. Since we want to make a point at a high level of generality, without a concrete model specification, we place no restrictions on stability. The efficient stable networks are of course constrained by the assumptions we’ve made on efficiency, and this tension with equality is sufficient for our impossibility result.

6.5.4. A general tension. The next result formalizes the general tension between efficient stable networks and equality by showing that, given the assumptions we’ve made, a network cannot be both Pareto efficient and regular—which, as argued above, is in general necessary but not sufficient for perfectly equitable outcomes.

Proposition 21. *Given Assumptions 19 and 20, there does not exist a Pareto-efficient and regular network.*

To help understanding of Proposition 21, we provide some intuition that follows the logic of the proof (see Section D of the Supplementary Appendix for the full proof). Pareto-efficient networks must have at least $n - 1$ links (by Assumption 19, all nodes must be in the same component) and a diameter of 2. This is because if there are $k \geq n - 1$ links then k pairs of the nodes are directly connected, and so the best possible path length distribution is for the other pairs of nodes to have path lengths of 2. This bound is achievable—consider any network that includes the star on n nodes as a subnetwork—and so by Assumption 19 must obtain. Given this, we can apply the Moore bound from graph theory. The Moore bound says that for any network component with diameter d and maximum degree no more than ψ , the number of nodes in the network, n , satisfies

$$n \leq 1 + \psi \frac{(\psi - 1)^d - 1}{\psi - 2}.$$

Thus for $d = 2$, in any regular network with degree r there must be $n \leq 1 + r^2$ nodes (equivalently, $r \geq \sqrt{n - 1}$). However, by Assumption 20 the average degree must be less than $\sqrt{n - 1}$, and so a Pareto-efficient regular network does not exist.

6.6. Timing of negotiations. A central assumption of our model is that agents first form the network of links that facilitate risk-sharing opportunities and only afterwards negotiate bilateral risk-sharing agreements with their neighbors. Our main motivation here is

that forming social connections occurs over a longer time horizon than negotiating or renegotiating risk-sharing arrangements with neighboring agents, so by the time the latter negotiations/renegotiations are underway, agents take the network as given. Although in our preferred interpretation these agreements are not binding, we assume that neighboring agents can enforce their bilateral agreements via implicit contracts: Not delivering a promised transfer ends the relationship. As formed relationships will provide surplus to the pair of agents and it takes considerable time to build new relationships, patient enough agents can credibly maintain informal risk-sharing agreements.⁵⁰

An alternative modeling assumption, that agents can sign binding agreements when they form relationships and that these agreements do not get renegotiated later, might be appropriate in some settings. A detailed investigation of such an alternative model is beyond the scope of the current paper, but in such a model it might not be the case that centrally located agents end up with a high share of total surplus, even if the network that forms is highly asymmetric, so it is not clear whether inequality will still arise endogeneously. It is possible that competition among agents (in the network formation phase) to become central will push down the payoffs of central agents to the level of the payoffs of peripheral agents linking with them. However, whether this is the case depends on the exact specification of the bargaining game, as can be demonstrated by a simple modification of our baseline model.

Assume that agents make binary risk-sharing agreements during the network formation stage. In particular, in the network formation game agents simultaneously indicate intentions to form a costly link with other agents, and at the same time propose bilateral state-contingent transfer agreements to each agent they approach—agreements that are conditional on the final network formed (that is, they propose a bilateral agreement for any possible network involving the current link). Allowing transfer agreements to be conditional on the final network is natural, as the values of the transfer agreement to the two agents depend on additional transfer arrangements with other agents. A link is formed if both agents indicate willingness to form it and they propose the same transfer agreement. Now for any network L , let $t^M(L)$ be a collection of bilateral risk-sharing agreements between neighboring agents in L that is Pareto efficient and allocates total surplus according to agents' Myerson values. In addition, let L^* be a pairwise-stable network in our original two-stage model. Then it is easy to establish that in the modified one-stage model such an outcome continues to satisfy a suitably modified version of our pairwise-stability condition. Suppose agents i and j indicate willingness to form a link if and only if $l_{ij} \in L^*$, and that for any L they propose transfer agreements $t_{ij}^M(L)$ and $t_{ji}^M(L)$. Then the agents will have exactly the same incentives to form an extra link or unilaterally drop a link as in our original model.⁵¹ Thus, the same surplus

⁵⁰An alternative interpretation is provided in Stole and Zwiebel (1996), which is more fitting in their intra-firm bargaining context: Agreements are formal contracts, but parties can hold each other up, and hence they can force renegotiation of contracts before the production phase.

⁵¹See the Supplementary Appendix for the formal arguments.

division (and same inequality) can prevail even when negotiations on risk sharing take place at the same time as network formation.

In the above game there are typically many other pairwise-stable outcomes that imply a different allocation of the surplus. A dynamic model of network formation and negotiations can potentially narrow down the set of possible payoff divisions, and this is an interesting direction to explore but it is outside the scope of the current paper. It is not clear to us if competition for central positions in such a model would eliminate inequality. It might be the case that initially there would be severe competition among the first few agents forming links to be central, but as a central agent emerges, she can leverage more and more bargaining power with newly arriving agents, as she becomes more central, and hence more desirable to link to relative to peripheral agents. We leave a formal investigation of these issues to future work.

The modified pairwise-stability concept allows only for coalitions of size two to form when deviating from a network-agreement. We have seen that this can be insufficient to eliminate the possibility of inequality. However, if larger coalitions are allowed to form in this context, coalitional deviations can potentially impose tighter constraints on the amount of inequality that can be sustained.

7. CONCLUSION

Our paper provides a relatively tractable model of endogenously formed networks and surplus division in a context of risk sharing that allows for heterogeneity in correlations between the incomes of pairs of agents. Such correlations have a sizeable impact on the potential of informal risk sharing to smooth incomes. We investigate the incentives for relationships that enable risk sharing to be formed, both within a group (caste or village) and across groups, giving access to less correlated income streams. We find a novel trade-off between equality and efficiency. Thus we identify new downsides to informal risk-sharing arrangements that can have important policy implications. This trade-off remains present in various generalizations and extensions of our baseline model.

Although we focus our analysis on risk sharing, our conclusions regarding network formation could apply in other social contexts too, as long as the economic benefits created by the social network are distributed similarly to the way they are in our model—a question that requires further empirical investigation.

Within the context of risk sharing, a natural next step would be to provide a dynamic extension of the analysis that allows for autocorrelation between income realizations. Another important direction in which this research agenda could be advanced is by studying a dynamic network problem (as discussed in Section 6.6).

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APPENDIX I. OMITTED PROOFS

Proof of Proposition 1. To prove the first statement, consider villagers' certainty-equivalent consumption. Let \hat{K} be some constant, and consider the certain transfer K' (made in all states of the world) that i requires to compensate her for keeping a stochastic consumption stream $c_i + \hat{K}$ instead of another stochastic consumption stream $c'_i + \hat{K}$:

$$\begin{aligned}
\mathbf{E}[v(c_i + \hat{K} + K')] &= \mathbf{E}[v(c'_i + \hat{K})] \\
-\frac{1}{\lambda} e^{-\lambda \hat{K}} e^{-\lambda K'} \mathbf{E}[e^{-\lambda c_i}] &= -\frac{1}{\lambda} e^{-\lambda \hat{K}} \mathbf{E}[e^{-\lambda c'_i}] \\
e^{\lambda K'} &= \frac{\mathbf{E}[e^{-\lambda c_i}]}{\mathbf{E}[e^{-\lambda c'_i}]} \\
(16) \quad K' &= \frac{1}{\lambda} \left(\ln \left(\mathbf{E}[e^{-\lambda c_i}] \right) - \ln \left(\mathbf{E}[e^{-\lambda c'_i}] \right) \right)
\end{aligned}$$

This shows that the amount K' needed to compensate i for taking the stochastic consumption stream $c_i + \hat{K}$ instead of $c'_i + \hat{K}$ is independent of \hat{K} . As a villager's certainty-equivalent consumption for a lottery is independent of her consumption level, certainty-equivalent units can be transferred among the villagers without affecting their risk preferences, and expected utility is transferable.

Next, we characterize the set of Pareto-efficient risk-sharing agreements. Borch (1962) and Wilson (1968) showed that a necessary and sufficient condition for a risk-sharing arrangement between i and j to be Pareto efficient is that in almost all states of the world $\omega \in \Omega := \mathbf{R}^{|S|}$,

$$(17) \quad \left(\frac{\partial v_i(c_i(\omega))}{\partial c_i(\omega)} \right) \bigg/ \left(\frac{\partial v_j(c_j(\omega))}{\partial c_j(\omega)} \right) = \alpha_{ij},$$

where α_{ij} is a constant. Substituting in the CARA utility functions, this implies that

$$\begin{aligned}
\frac{e^{-\lambda c_i(\omega)}}{e^{-\lambda c_j(\omega)}} &= \alpha_{ij} \\
c_i(\omega) - c_j(\omega) &= -\frac{\ln(\alpha_{ij})}{\lambda} \\
\mathbf{E}[c_i(\omega)] - \mathbf{E}[c_j(\omega)] &= -\frac{\ln(\alpha_{ij})}{\lambda} \\
(18) \quad c_i(\omega) - c_j(\omega) &= \mathbf{E}[c_i(\omega)] - \mathbf{E}[c_j(\omega)]
\end{aligned}$$

Letting i and j be neighbors such that $j \in \mathbf{N}(i)$, equation (18) means that when i and j reach any Pareto-efficient risk-sharing arrangement their consumptions will differ by the same constant in all states of the world. Moreover, by induction the same must be true for all pairs of path-connected villagers.

Consider now the problem of splitting the incomes of a set of villagers \mathbf{S} in each state of the world to minimize the sum of their consumption variances:

$$(19) \quad \min_{\mathbf{c}} \sum_{i \in \mathbf{S}} \text{Var}(c_i) \quad \text{subject to} \quad \sum_{i \in \mathbf{S}} y_i(\omega) = \sum_{i \in \mathbf{S}} c_i(\omega) \quad \text{for all } \omega.$$

If we denote the CDF of income probability distribution on $\Omega = \mathbf{R}^{|\mathbf{S}|}$ by $F(\cdot)$, then

$$(20) \quad \sum_{i \in \mathbf{S}} \text{Var}(c_i) = \int_{\Omega} \sum_{i \in \mathbf{S}} (c_i(\omega) - \mathbf{E}[c_i])^2 dF(\omega).$$

Since $\text{Var}(c_i(\omega) + a_i) = \text{Var}(c_i(\omega))$, the sum of variances is invariant to state-independent changes in a consumption profile, and the variance-minimizing consumption profile exists for any profile of expected consumptions $\{\mathbf{E}[c_i]\}_{i \in \mathbf{S}}$: $\sum_{i \in \mathbf{S}} \mathbf{E}[c_i] = \sum_{i \in \mathbf{S}} \mathbf{E}[y_i]$. Fix any such profile of expected consumptions, $\{\mathbf{E}[c_i]\}_{i \in \mathbf{S}}$. Similarly to Wilson (1968), we apply Theorem 1 from Zahl (1963) to our minimization problem. We denote a Lagrange multiplier attached to the constraint $\sum_{i \in \mathbf{S}} y_i(\omega) = \sum_{i \in \mathbf{S}} c_i(\omega)$ by $\gamma(\omega)$. Then the corresponding Lagrangian of the problem is

$$\int_{\Omega} \left[\sum_{i \in \mathbf{S}} (c_i(\omega) - \mathbf{E}[c_i])^2 - \gamma(\omega) \sum_{i \in \mathbf{S}} c_i(\omega) \right] dF(\omega).$$

By pointwise minimization with respect to $c_i(\omega)$, we obtain that for each $i \in \mathbf{S}$ and almost every $\omega \in \Omega$, $2(c_i^*(\omega) - \mathbf{E}[c_i]) = \gamma(\omega)$. Thus $c_i^*(\omega) - c_j^*(\omega) = \mathbf{E}[c_i(\omega)] - \mathbf{E}[c_j(\omega)]$ for all $i, j \in \mathbf{S}$. Note that this equality implies that $\mathbf{E}[c_i^*(\omega)] = \mathbf{E}[c_i]$, and $\{c_i^*(\omega)\}$ indeed solves the minimization problem. Thus, the condition $c_i^*(\omega) - c_j^*(\omega) = \mathbf{E}[c_i(\omega)] - \mathbf{E}[c_j(\omega)]$ for almost all ω is exactly the same as the necessary and sufficient condition for an ex-ante Pareto efficiency. Therefore, a risk-sharing agreement is Pareto efficient if and only if the sum of the consumption variances for all path-connected villagers is minimized.

Using the necessary and sufficient condition for efficient risk sharing, we obtain

$$(21) \quad \sum_{k \in \mathbf{S}} y_k(\omega) = \sum_{k \in \mathbf{S}} c_k(\omega) = |\mathbf{S}| c_i(\omega) - \sum_{k \in \mathbf{S}} (\mathbf{E}[c_i(\omega)] - \mathbf{E}[c_k(\omega)]),$$

which implies that

$$(22) \quad c_i(\omega) = \frac{1}{|\mathbf{S}|} \sum_{k \in \mathbf{S}} y_k(\omega) + \frac{1}{|\mathbf{S}|} \sum_{k \in \mathbf{S}} (\mathbf{E}[c_i(\omega)] - \mathbf{E}[c_k(\omega)]) = \frac{1}{|\mathbf{S}|} \sum_{k \in \mathbf{S}} y_k(\omega) + \tau_i,$$

where $\tau_i = \mathbf{E}[c_i(\omega)] - \mathbf{E}[\frac{1}{|\mathbf{S}|} \sum_{k \in \mathbf{S}} y_k(\omega)]$. □

Proof of Lemma 4. Agent i 's net benefit from forming link l_{ij} is $(MV_i(L) - MV_i(L \setminus \{l_{ij}\}) - \kappa_w)$. We need to show that

$$(23) \quad MV_i(L) - MV_i(L \setminus \{l_{ij}\}) = MV_j(L) - MV_j(L \setminus \{l_{ij}\}) = md(i, j, L)V.$$

Some additional notation will be helpful. Suppose agents arrive in a random order, with a uniform distribution on all possible arrival orders. The random variable $\widehat{\mathbf{S}}_i \subseteq \mathbf{N}$ identifies the set of agents, including i , who arrive weakly before i . For each arrival order, we then have an associate network $\mathcal{L}_L(\widehat{\mathbf{S}}_i)$ that describes the network formed upon i 's arrival (the subnetwork of L induced by the agents in $\widehat{\mathbf{S}}_i$). Let $q(i, j, L)$ be the probability that i and j are path connected on network $\mathcal{L}_L(\widehat{\mathbf{S}}_i)$.

The certainty-equivalent value of the reduction in variance due to a link l_{ij} in a network $\mathcal{L}_L(\widehat{\mathbf{S}}_i)$ is V if the link is essential and 0 otherwise. The change in i 's Myerson value, $MV_i(L) - MV_i(L \setminus \{l_{ij}\})$, is then $(q(i, j, L) - q(i, j, L \setminus \{l_{ij}\}))V$. However, $q(i, j, L) = 1/2$. To see this, note that $l_{ij} \in L$, and therefore in every order of arrival in which i arrives after j (which happens with probability $1/2$), i and j are path connected on the network $\mathcal{L}_L(\widehat{\mathbf{S}}_i)$, while i and j are never path connected on $\mathcal{L}_L(\widehat{\mathbf{S}}_i)$ if j arrives after i .

The probability $q(i, j, L \setminus \{l_{ij}\})$ can be computed by the inclusion–exclusion principle, using the fact that the probability of a path connecting i and j existing on network $\mathcal{L}_{L \setminus \{l_{ij}\}}(\widehat{\mathbf{S}}_i)$ is equal to the probability that for some path connecting i and j on $L \setminus \{l_{ij}\}$ all agents on the path are present in $\widehat{\mathbf{S}}_i$. Thus

$$(24) \quad q(i, j, L \setminus \{l_{ij}\}) = \sum_{k=1}^{|\mathbf{P}(i, j, L \setminus \{l_{ij}\})|} (-1)^{k+1} \left(\sum_{1 \leq i_1 < \dots < i_k \leq |\mathbf{P}(i, j, L)|} \left(\frac{1}{|P_{i_1} \cup \dots \cup P_{i_k}|} \right) \right).$$

We therefore have that

$$(25) \quad MV_i(L) - MV_i(L \setminus \{l_{ij}\}) = (1/2 - q(i, j, L \setminus \{l_{ij}\}))V = md(i, j, L)V,$$

where the last equality follows from the definition of Myerson distance. \square

Proof of Proposition 6. Part (i): By Remark 3 and under our regularity condition, all efficient networks are tree networks. By definition, in all tree networks any pair of agents i and j have a unique path between them. Thus for a tree network L with diameter $d(L)$, there exist agents i and j with a unique path between them of length $d(L)$ and all other pairs of agents have a weakly shorter path between them. Thus by equation (11):

$$(26) \quad md(i, j, L) = \frac{1}{2} - \frac{1}{d(L)} \geq md(k, k', L) \quad \text{for all } k, k' \in \mathbf{N}.$$

By Proposition 5, there is no underinvestment in any stable network. Lemma 4 therefore implies that the efficient network L is stable if and only if $md(k, k', L) \leq \kappa_w/V$ for all k, k' such that $l_{kk'} \notin L$. As $md(i, j, L) \geq md(k, k', L)$ and $md(i, j, L) = 1/2 - 1/d(L)$ (see equation (26)), this condition simplifies and the efficient network L is stable if and only if

$$(27) \quad \frac{V - 2\kappa_w}{V} \leq \left(\frac{2}{d(L)} \right).$$

As $d(L)$ gets large, the right-hand side converges from above to 0, and so in the limit the condition for stability becomes $V \leq 2\kappa_w$, which is violated by our regularity condition. Thus there exists a finite $\bar{d}(L)$ such that the efficient network L is stable if and only if $d(L) \leq \bar{d}(L)$.

Rearranging equation (27), L is stable if and only if

$$(28) \quad d(L) \leq 2 \left(\frac{V}{V - 2\kappa_w} \right).$$

Thus the key threshold is $\bar{d}(\kappa_w) = \lfloor 2V/(V - 2\kappa_w) \rfloor$.

Fixing the number of agents $|\mathbf{N}|$ in an efficient (tree) network L , the star network is the unique (tree) network (up to a relabeling of players) that minimizes the diameter $d(L)$, while the line network is the unique (tree) network (up to a relabeling of players) that maximizes the diameter $d(L)$. The result now follows immediately.

Part (ii): On any efficient network, all links are essential and generate a net surplus of $V - 2\kappa_w > 0$, where the inequality follows from our regularity condition. As i and j must benefit equally at the margin from the link l_{ij} (see condition (ii) in the definition of agreements that are robust to split-the-difference renegotiations), agent i 's expected payoff on an efficient network L is

$$(29) \quad u_i(L) = |\mathbf{N}(i; L)|(V/2 - \kappa_w) > 0.$$

Thus i 's net payoff is proportional to her degree.

For any tree network L other than the star network, let agent k be one of the agents with the highest degree. Consider a link $l_{ij} \in L$ such that $i, j \neq k$. As L is a tree, there is a unique path from i to k and a unique path from j to k . As we are on a tree network, either the path from j to k passes through i or the path from i to k passes through j . Hence either i or j is closer to k , and without loss of generality we let i have a longer path to k than j . We now delete the link l_{ij} and replace it with the link l_{ik} . This operation generates a new tree network. Moreover, repeating this operation until there are no links l_{ij} such that $i, j \neq k$ defines an algorithm.

This algorithm terminates at a star network as in a star network there are no links l_{ij} such that $i, j \neq k$. Moreover the operation can be applied to any other tree network because on all other tree networks there exists an l_{ij} such that $i, j \neq k$. Finally, in each step of the algorithm the degree of k increases, and so the algorithm must terminate in a finite number of steps. Moreover, the algorithm must terminate at the star network with k at the center.

By construction, at each step of the above algorithm we decrease the degree of some agent $j \neq k$ and increase the degree of k . Suppose we start with a network L and consider a step in this rewiring where the link l_{ij} is deleted and replaced by the link l_{ik} . Only the expected payoffs of agents j and k on L and $L \cup \{l_{ik}\} \setminus \{l_{ij}\}$ change. The degrees of all other agents remain constant, and hence by equation (29) so do their payoffs. Letting $\alpha = (V/2 - \kappa_w)$, we have $u_j(L) = \alpha d_j(L)$, $u_k(L) = \alpha d_k(L)$, $u_j(L \cup \{l_{ik}\} \setminus \{l_{ij}\}) = \alpha(d_j(L) - 1)$ and $u_k(L \cup \{l_{ik}\} \setminus \{l_{ij}\}) = \alpha(d_k(L) + 1)$.

It follows that welfare $W(u) = \sum_i f(u_i)$ (see equation (9)) decreases through the rewiring in this step if and only if

$$(30) \quad f(\alpha(d_j - 1)) + f(\alpha(d_k + 1)) - f(\alpha d_j) - f(\alpha d_k) < 0,$$

which is equivalent to

$$(31) \quad f(\alpha(d_k + 1)) - f(\alpha d_k) < f(\alpha d_j) - f(\alpha(d_j - 1))$$

As $f(\cdot)$ is increasing, strictly concave, and differentiable, $f'(\alpha d_j)\alpha < f(\alpha d_j) - f(\alpha(d_j - 1))$ and $f'(\alpha d_k)\alpha > f(\alpha(d_k + 1)) - f(\alpha d_k)$. Moreover, by concavity, $f'(\alpha d_j) \geq f'(\alpha d_k)$ (as $d_k \geq d_j$). Combining these inequalities establishes the claim that $f(\alpha(d_k + 1)) - f(\alpha d_k) < f(\alpha d_j) - f(\alpha(d_j - 1))$.

Thus at each step in the rewiring, welfare $W(u)$ decreases. For each network L' reached during the algorithm, we can consider the average expected utility $u'(L')$ which if distributed equally would generate the same level of welfare as that obtained on L . As aggregate welfare is decreasing at each step in the rewiring, $u'(L)$ must be decreasing too. However, the total surplus generated by risk sharing remains constant, and so the average expected utility \bar{u} remains constant. Recall that Atkinson's inequality measure/index is given by $I(L) = (1 - (u'(L)/\bar{u}))$. Thus at each step in the rewiring the inequality measure $I(L)$ increases. As this rewiring can be used to move from any tree network to the star network, stars network and only star networks maximize inequality among the set of tree networks (which corresponds to the set of efficient networks under our regularity condition). As this argument holds for any strictly increasing, concave, and differentiable function f , it holds for all inequality measures in the Atkinson class.

Consider now an alternative rewiring of a tree network L . Let k be one of the agents with highest degree on L , and let j be one of the agents with degree 1 on L . As tree networks contain no cycles, there can always exist agents with degree 1 (leaf agents). Pick one of k 's neighbors $i \in \mathbf{N}(k; L)$ such that $i \neq j$, remove the link l_{ik} from L , and add the link l_{ij} to L . This operation generates a new tree network. Repeating this operation until the highest-degree agent has degree 2 defines an algorithm. As the unique tree network with a highest degree of 2 is the line network, the algorithm terminates at line networks and only at line networks. At each stage in the rewiring, we either reduce the degree of the highest-degree agent or reduce the number of agents who have the highest degree. Thus the algorithm must terminate in a finite number of steps and at a line network. Moreover, reversing the argument above, inequality is reduced at each step of the rewiring for any inequality measure in the Atkinson class. \square

Proof of Proposition 7. By definition, within-group underinvestment for a network L requires that there exist an $l_{ij} \notin L$ such that $G(i) = G(j)$ and for which $TS(L \cup \{l_{ij}\}) - TS(L) > 2\kappa_w$.

As $TS(L \cup \{l_{ij}\}) - TS(L) = 0$ for all non-essential links, l_{ij} must be essential on $L \cup \{l_{ij}\}$. Thus l_{ij} is also essential on $\widehat{L} \cup \{l_{ij}\}$ for any $\widehat{L} \subseteq L$. Equation (13) then implies that $TS(\widehat{L} \cup \{l_{ij}\}) - TS(\widehat{L}) \geq V$ for any $\widehat{L} \subseteq L$.

Consider any arrival order in which i arrives after j , and let \mathbf{S}_i be the set of agents that arrive (strictly) before i . When i arrives, her marginal contribution to total surplus without l_{ij} when i arrives is then $TS(L(\mathbf{S}_i \cup \{i\})) - TS(L(\mathbf{S}_i))$, while with l_{ij} it is $TS(L(\mathbf{S}_i \cup \{i\}) \cup \{l_{ij}\}) - TS(L(\mathbf{S}_i))$. Thus i 's additional marginal contribution to total surplus when l_{ij} has been formed is $TS(L(\mathbf{S}_i \cup \{i\}) \cup \{l_{ij}\}) - TS(L(\mathbf{S}_i \cup \{i\}))$. As $L(\mathbf{S}_i \cup \{i\}) \subseteq L$, by the above argument $TS(L(\mathbf{S}_i \cup \{i\}) \cup \{l_{ij}\}) - TS(L(\mathbf{S}_i \cup \{i\})) \geq V$. As i arrives after j in half of the arrival orders, i 's average additional incremental contribution to total surplus when l_{ij} has been formed is at least $V/2$. Thus $MV_i(L \cup \{l_{ij}\}) - MV_i(L) \geq V/2$. An equivalent argument establishes that $MV_j(L \cup \{l_{ij}\}) - MV_j(L) \geq V/2$. Under our regularity condition, $V/2 > \kappa_w$, and so i and j have a profitable deviation to form l_{ij} and the network L is not stable. As L was an arbitrary network with within-group underinvestment, there is no stable network with within-group underinvestment. \square

Proof of Proposition 8. The proof of the first part of the statement has four steps.

Step 1: Consider any efficient network L that is robust to within-group overinvestment inefficiency. This implies that for all path connected agents i, j such that $G(i) = G(j)$ and $l_{ij} \notin L$, either $MV_i(L \cup \{l_{ij}\}) - MV_i(L) \leq \kappa_w$ or $MV_j(L \cup \{l_{ij}\}) - MV_j(L) \leq \kappa_w$. However, by condition (i) in the definition of agreements that are robust to split-the-difference renegotiations, $MV_i(L \cup \{l_{ij}\}) - MV_i(L) = MV_j(L \cup \{l_{ij}\}) - MV_j(L)$, and so both $MV_i(L \cup \{l_{ij}\}) - MV_i(L) \leq \kappa_w$ and $MV_j(L \cup \{l_{ij}\}) - MV_j(L) \leq \kappa_w$.

Step 2: Let a network $\widehat{L} := \{l_{ij} : G(i) = G(j), l_{ij} \in L\}$ be formed from L by deleting all across-group links. Consider any subset of agents $\mathbf{S} \subseteq \mathbf{N}$ such that $i, j \in \mathbf{S}$. As the network L is efficient, it is a tree network that minimizes the number of across-group links conditional on a given set of agents being in a component. This implies that the unique path between i and j cannot contain an across-group link. Therefore, i is path connected to j on the induced subnetwork $L(\mathbf{S})$ if and only if i is path connected to j on the induced subnetwork $\widehat{L}(\mathbf{S})$. Thus by equation (13), the additional variance reduction that i and j can now achieve by forming a superfluous across-group link on $\widehat{L}(\mathbf{S})$ is weakly lower than on $L(\mathbf{S})$. Therefore, by the definition of the Myerson value (equation (6)), $MV_i(\widehat{L} \cup \{l_{ij}\}) - MV_i(\widehat{L}) \leq MV_i(L \cup \{l_{ij}\}) - MV_i(L)$ and $MV_j(\widehat{L} \cup \{l_{ij}\}) - MV_j(\widehat{L}) \leq MV_j(L \cup \{l_{ij}\}) - MV_j(L)$. This implies that \widehat{L} is robust to within-group overinvestment.

Step 3: Let a network \widehat{L}' be formed from \widehat{L} by rewiring (alternately deleting and then adding a link) each within-group network into a star (for an algorithm that does this, see part (ii) of the proof of Proposition 6). Consider any two agents i', j' such that $G(i') = G(j')$ and $l_{i'j'} \notin \widehat{L}'$. By part (i) of Proposition 6, $MV_{i'}(\widehat{L}' \cup \{l_{i'j'}\}) - MV_{i'}(\widehat{L}') \leq MV_{i'}(\widehat{L} \cup \{l_{ij}\}) - MV_{i'}(\widehat{L})$

and $MV_{j'}(\widehat{L}' \cup \{l_{i'j'}\}) - MV_{j'}(\widehat{L}') \leq MV_j(\widehat{L} \cup \{l_{ij}\}) - MV_j(\widehat{L})$. Thus \widehat{L}' is robust to within-group overinvestment.

Step 4: Finally, consider any network $L' \in \mathcal{L}^{CSS}$. This network can be formed by adding a set of across-group links to a network \widehat{L}' such that $\widehat{L}' \subseteq L'$ and if $l_{kk'} \in L' \setminus \widehat{L}'$ then $G(k) \neq G(k')$. Consider any subset of agents $\mathbf{S}' \subseteq \mathbf{N}$ such that $i', j' \in \mathbf{S}'$. Recall that $G(i') = G(j')$, and note that by the construction of L' , $l_{i'j'} \notin L'$. On the induced subnetwork $L'(\mathbf{S}')$, either i' is path connected to j' , in which case $l_{i'j'}$ would be superfluous if added, or i' and j' are isolated nodes. This is because the within-group network structure for group $G(i')$ is a star. Thus whenever $l_{i'j'}$ would not be superfluous, the change in i' and j' 's Myerson value if it were added is independent of the across-group links that are present: $MV_{i'}(L' \cup \{l_{i'j'}\}) - MV_{i'}(L') = MV_{i'}(\widehat{L}' \cup \{l_{i'j'}\}) - MV_{i'}(\widehat{L}')$ and $MV_{j'}(L' \cup \{l_{i'j'}\}) - MV_{j'}(L') \leq MV_{j'}(\widehat{L}' \cup \{l_{i'j'}\}) - MV_{j'}(\widehat{L}')$. Thus L' is robust to within-group overinvestment.

We turn now to the second part of the result. If $L \notin \mathcal{L}^{CSS}$, then there will be agents i, j such that $G(i) = G(j)$ and $l_{ij} \notin L$ such that either the within-group network structure for $G(i)$ is not a star, or it is a star but there are across-group links being held by an agent who is not the center agent. In the first case, the inequality in step 3 will be strict by Proposition 6. In the second case, without loss of generality we can let agent i be the non-center agent holding the across-group link. Then by equation (13), the inequality in step 2 will be strict. Thus for some parameter values, L will not be robust to within-group overinvestment, but L' will be. \square

Proof of Lemma 9. Denote the set of all possible arrival orders for the set of agents \mathbf{N} by $\mathcal{A}(\mathbf{N})$. Order this set of $|\mathbf{N}|!$ arrival orders in any way, denoting the k th arrival order by $\widehat{A}_k \in \mathcal{A}(\mathbf{N})$. We will then construct an alternative ordering, in which we denote the k th arrival order by $\widetilde{A}_k \in \mathcal{A}(\mathbf{N})$, such that for arrival order \widetilde{A}_k ,

- (i) i arrives at the same time as agent i' does for the arrival order \widehat{A}_k ;
- (ii) when i arrives, she connects to exactly the same set of agents from $\mathbf{N} \setminus \mathbf{S}_0$ that i' connects to upon her arrival for the arrival order \widehat{A}_k ;
- (iii) when i arrives, she connects to weakly more agents from \mathbf{S}_0 than i' connects to upon her arrival for the arrival order \widehat{A}_k .

Equation (15) shows that the risk reduction, and hence the marginal contribution made by an agent $k \in \mathbf{S}_0$ from providing the across-group link l_{kj} is an increasing function of the number of agents in k 's component who are also in her group. It then follows that

$$(32) \quad MV(i; L \cup \{l_{ij}\}) - MV(i; L) > MV(i'; L \cup \{l_{i'j}\}) - MV(i'; L).$$

To construct the alternative ordering of the set $\mathcal{A}(\mathbf{N})$ as claimed, we will directly adjust individual arrival orders but in a way that preserves the set $\mathcal{A}(\mathbf{N})$. First, for each arrival order, we switch the arrival positions of i' and i . This alone is enough to ensure that conditions (i)

and (ii) are satisfied. There are $|\mathbf{S}_0|!$ possible arrival orders for the set of agents \mathbf{S}_0 . Ignoring for now the other agents, we label these arrival orders lexicographically. First, we order them in ascending order by when i arrives. Next, we order them in ascending order by the number of agents i is connected to upon her arrival. Breaking remaining ties in any way, we have labels $1_i, 2_i, \dots, |\mathbf{S}_0|_i$. We then let every element of $\mathcal{A}(N)$ inherit these labels, so that two arrival orders receive the same label if and only if the agents in \mathbf{S}_0 arrive in the same order. We now construct a second set of labels by doing the same exercise for i' , and denote these labels by $1_{i'}, 2_{i'}, \dots, |\mathbf{S}_0|_{i'}$. We are now ready to make our final adjustment to the arrival orders. For each original arrival order \hat{A}_k , we find the associated (second) label. Suppose this is $x_{i'}$. We then take the current k th arrival order (given the previous adjustment) and reorder (only) the agents in \mathbf{S}_0 , so that the newly constructed arrival order now has (first) label x_i . Because of the lexicographic construction of the labels, the arrival position of agent i will not change as a result of this reordering of the arrival positions of agents in \mathbf{S}_0 , so conditions (i) and (ii) are still satisfied. In addition, condition (iii) will now be satisfied by the definition of i being more central than i' . The only remaining thing to verify is that the set of arrival orders we are considering has not changed (i.e., that we have, as claimed, constructed an alternative ordering of the set $\mathcal{A}(N)$), and this holds by construction. \square

Proof of Proposition 10. Let L be an efficient network that is robust to across-group under-investment. This implies that for any across-group link $l_{ij} \in L$ between groups $g = G(i)$ and $\hat{g} = G(j) \neq g$, $MV_i(L) - MV_i(L \setminus \{l_{ij}\}) = MV_j(L) - MV_j(L \setminus \{l_{ij}\}) \geq \kappa_a$, which follows from condition (i) in the definition of robustness to split-the-difference renegotiations.

We now rewire L . As the network L is efficient, it is a tree network that minimizes the number of across-group links conditional on a given set of agents being in a component. This implies that the unique path between any two agents from the same group cannot contain an across-group link. We can therefore rewire the within-group network structures of L to obtain a star by sequentially deleting and then adding within-group links (an algorithm that does this is presented in the proof of part (ii) of Proposition 6). We do this rewiring so that agent i is the center agent of the within-group network for group $G(i)$, and j is the center agent of the within-group network for group $G(j)$. Finally, we rewire across-group links so that the same groups remain directly connected but all across-group links are held by the center agents. Let the network obtained be L' . By construction, $L' \in \mathcal{L}^{CCS}$.

Under our definition of Myerson centrality, it is straightforward to verify that both i and j are weakly more Myerson central within their respective groups on network L' than on network L . An argument almost identical to that in the proof of Lemma 9 then implies that i and j have better incentives to keep the link l'_{ij} on L' than i and j have to keep the link l_{ij} on L (because the argument is more or less identical, we skip it). Thus,

$$(33) \quad MV_i(L') - MV_i(L' \setminus \{l'_{ij}\}) \geq MV_i(L) - MV_i(L \setminus \{l_{ij}\})$$

$$(34) \quad MV_j(L') - MV_j(L' \setminus \{l'_{ij}\}) \geq MV_j(L) - MV_j(L \setminus \{l_{ij}\})$$

Network L' is therefore robust to across-group underinvestment. Moreover, by Lemma 9 whenever the within-group networks of i and j on network L are not both stars with i and j at the centers, the inequality is strict because both i and j are strictly more Myerson central within their respective groups on L' than on L . There then exists a range of parameter specifications for which any center-connected star network $L' \in \mathcal{L}^{CCS}$ is robust to across-group underinvestment but L is not. \square