Gradual bidding in eBay-like auctions*

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This paper shows that in online auctions like eBay, if bidders can only place bids at random times, then many different equilibria arise besides truthful bidding, despite the option to leave proxy bids. These equilibria can involve gradual bidding, periods of inactivity, and waiting to start bidding towards the end of the auction - bidding behaviors common on eBay. Bidders in such equilibria implicitly collude to keep the increase of the winning price slow over the duration of the auction. In a common value environment, we characterize a class of equilibria that include the one in which bidding at any price is maximally delayed, and all bids minimally increment the price. The seller's revenue can be a small fraction of what could be obtained at a sealed-bid second-price auction. With many bidders, we show that this equilibrium has the feature that bidders are passive until near the end of the auction, and then they start bidding incrementally.

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1 Introduction

A distinguishing feature of online auctions, relative to spot auctions, is that they typically last a relatively long time.[1] However, this aspect is often suppressed in the related economics literature. In particular, if bidders have private valuations, online auction mechanisms such as eBay's, in which bidders can leave a proxy bid and the highest bidder wins the object at a price equal to the second highest bid (plus a minimum bid increment), are commonly regarded as strategically equivalent to second-price sealed-bid auctions. Since bidding one's true valuation in the latter context is a weakly dominant strategy, and placing a bid takes some effort, the above arguments suggest that a rational bidder at an eBay-like auction should only place one bid, equal to her true valuation, at her earliest convenience.

In contrast with the above predictions, observed bidding behavior on eBay involves substantial gradual bidding and last-minute bidding (commonly referred to as "sniping"). Ockenfels and Roth (2006) report that the average number of bids per bidder is 1.89 and 38% of bidders submit more than one bid. In a field experiment by Hossain and Morgan (2006), 76% of the auctions had at least one bidder placing multiple bids.[2] Regarding sniping, Roth and Ockenfels (2002) report that 18% of auctions in their data received bids in the last minute, while Bajari and Hortacsu (2003) find that the median winning bid arrives after 98.3% of the auction time elapsed, while 25% of the winning bids arrive after 99.8% of the auction time elapsed.

While Roth and Ockenfels (2002) propose a model in which last-minute bidding can be an equilibrium,[3] the existing literature (including Roth and Ockenfels (2002)) typically considers gradual bidding to be a naive (irrational) behavior. Relatedly, Ku et al. (2005) explain bidding behavior in online auctions with a model of emotional decision-making and competitive arousal, Ely and Hossain (2009) describe incremental bidders as confused, mistaking eBay's proxy system for an ascending auction, while Hossain (2008) posits behavioral buyers who learn about their own valuations through the process of placing bids.[4]

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[1] On eBay, sellers can specify durations from 1 to 30 days.

[2] See also Zeithammer and Adams (2010), who find evidence of incremental bidding in online auction field data. For example they find that the frequency of the winning bid being exactly the minimum increment higher than the runner up bid to be too high than what would be implied by truthful bidding (they observe the winning bidder's proxy bid, hence they can perform this test). Moreover, such an event is much more likely if the winning bidder places a bid after the runner up bidder than vice versa.

[3] However, Hasker et al. (2009), using ebay data, reject the hypothesis that bidders follow a war of sniping profile as in Roth and Ockenfels (2002). See also Ariely et al. (2005) for a related laboratory experiment.

[4] See also Compte and Jehiel (2004) and Rasmusen (2006) for bidders learning their true valuations during the auction. Other explanations include the presence of multiple overlapping auctions for identical or close substitute objects as in Peters and Severinov (2006), Hendricks et al. (2009), and Fu (2009). However, gradual and last-minute bidding seems to be prevalent for rare or unique objects, too, not only for objects with many close substitutes being auctioned at any time (for example, they occur in the experiments of Ariely et al. (2005) despite there is no concurrent competing auction). Furthermore, as Hossain (2008) points out, this type of argument also has trouble explaining many bids by the same bidder in a short interval of

In this paper we show that, in a private value context, if bidders are not present during the entire auction (a clearly unrealistic scenario for online auctions) and instead have periodic random opportunities to check the auction's status and place bids,[5] then there can be many different equilibria of the resulting game with perfectly rational bidders, despite the possibility of proxy bids. The best equilibrium for the seller in this game still implies truthful bidding, upon the first bidding opportunity. If the time horizon of the auction is long, the seller's revenue in this equilibrium is approximately what he could get in a second-price sealed bid auction. However, there are typically many other equilibria, in weakly undominated strategies, which imply incremental bidding, long periods of intentionally not placing bids, and potential sniping. The seller's expected revenue from these equilibria can be a very small fraction of the expected revenue from the best equilibrium, even when the auction's time horizon is arbitrarily large and bidding opportunities are frequent.

To understand the intuition for the existence of such equilibria, consider two bidders, each with valuation $v > 2$ where bidding opportunities (including potential proxy bids) follow some random arrival process. Suppose that the initial price is 0. Clearly, there is an equilibrium in which whenever the current price is below v and a bidder who has the opportunity places a bid of v. However, if the time horizon is hort, there is another equilibrium in which a bidder, when she gets the chance, increases the price only by the minimum increment. The key insight here is that gradual bidding is a self-enforcing form of implicit collusion: if other bidders follow such a strategy then it is strictly in the interest of a bidder to do likewise. Increasing the price by more than the minimum increment does not increase the likelihood of eventually winning the object, only speeds up the increase of the leading price, reducing the winning bidder's surplus.

If the time horizon of the auction is long (relative to the arrival rates) then, besides gradual bidding, it also becomes optimal for bidders to pass on bidding opportunities, for prolonged periods of time. In particular, we show that for certain prices bidders pass on bidding opportunities before a cutoff point in time, and only start incrementing bids after the cutoff. For this reason, the seller's expected revenue can be a small fraction of v, no matter how long the auction, or how frequently bidding opportunities arise. This also means that in dynamic auction environments where bidders cannot be present for the whole duration of the auction, the classic result of Bulow and Klemperer (1996), that for a given set of buyers a seller can guarantee a large share of possibly attainable payoff by simply running an ascending price auction, no longer holds.

Another noteworthy feature of a gradual-bidding equilibrium is that it can prescribe placing the minimum bid upon the first arrival, and then a long period of inactivity, followed by all bidders trying to incrementally outbid each other

time, which is quite common in ebay. Bajari and Hortacsu (2003) raise the possibility that all ebay auctions have some common value component.

[5] It is important for our results that whenever a bidder gets the chance to check the status of the auction, she can place multiple bids. In particular, if she places a bid incrementing the current price but gets notified that this bid was not enough to take over the lead, she can place a subsequent higher bid.

towards the end of the auction. This feature is consistent with the finding that the time-profile of bids is bimodal, with a small peak near the beginning and a large peak near the end (Roth and Ockenfels (2002), Bajari and Hortacsu (2003)).

For any number of bidders and arbitrary time horizon, we characterize a class of equilibria in which overbidding at any price is maximally delayed by the threat of play switching to a truthful equilibrium if anyone places an earlier bid. This equilibrium is non-Markovian if the time horizon is long, and has the feature that along the equilibrium path players wait until the end of the auction (with bids placed earlier triggering switching to a truthful bidding equilibrium) and then bid gradually. This bidding behavior is similar to sniping, as in Roth and Ockenfels (2002), with the difference that the snipers only overbid incrementally instead of truthfully. In fact, it can be shown that in our framework Roth and Ockenfels type strategy profiles, in which bidders wait until the end of the auction and then bid truthfully, cannot be an equilibrium. Such profiles in a continuous-time framework (with no special "last period," as in Roth and Ockenfels) unravel, with each bidder wanting to start sniping at least a little earlier than the others.

Among these maximally delayed equilibria, we focus on the one in which, along the equilibrium path, the winning price increases completely gradually (at each step only by the minimum bid increment), as this is the equilibrium with the lowest expected revenue for the seller. We show that the seller's expected revenue in this worst case scenario, somewhat paradoxically, decreases in the value of the object. The basic intuition is that the threat of reversion to a truthful equilibrium can induce players to wait longer before starting to place bids when the value of the object is high, which counteracts the effect that conditional on bidders being active, the expected winning price is higher when the value of the object or the number of bidders increases. We show that the first effect in fact dominates the second one. This implies that for very highly valued objects, in the seller's worst equilibrium expected revenue becomes a vanishing fraction of the object's value. In contrast, for a fixed value of the object, taking the number of bidders to infinity implies that even in the seller's worst equilibrium, expected revenue converges to the value of the object. Numerical computations however show that the convergence can be slow and even for 20 bidders, the seller's expected revenue can be a small fraction of the object's value. For this reason, if there are search frictions that make it impossible for all potentially interested buyers to actively participate in online auctions, even if the increased buyer surplus attracts some new bidders to such auctions, the expected revenue of a seller in an eBay-like auction might only be a small fraction of the object's true value.

For short time horizons with any number of bidders, and for arbitrary time horizons with many bidders, we also characterize a class of Markovian equilibria with gradual bidding, including one with completely gradual bidding.[6] In the

[6]For arguments in favor of focusing on Markov perfect equilibria in asynchronous-move games similar to the one we consider, see Bhaskar and Vega-Redondo (2002) and Bhaskar et

case of many bidders, these equilibria also have the feature that bidders wait until near the end of the auction and then start bidding gradually. We also show that for any number of bidders and time horizon, there exist Markovian equilibria with some gradual bidding.to any time-dependent arrival process for which arrival rates remain bounded, including

For analytical convenience, for most of the analysis we assume that arrival rates follow a time-independent Poisson arrival process. However, we show that the qualitative conclusions of the paper, such as the existence of gradual bidding equilibria, extend to any time-dependent arrival process for which arrival rates stay bounded,[7] including ones for which arrival rates steeply increase towards the end of the auction. Relative to the case of constant arrival rates, in such scenarios players wait longer before they start bidding, and (gradual) bidding might only take place for a short interval before the auction ends. For this reason, even when arrival rates are very high before the end of the auction, as long as they are bounded from above, the expected number of bids and the winning price can remain very low.

We also demonstrate, in a context with two possible valuations, that the existence of gradual bidding equilibria extends beyond the complete information symmetric bidders case, to specifications in which different bidders can have different valuations, or are uncertain about other bidders' valuations.

Our work is part of a recent string of papers examining continuous time games with random discrete opportunities to take actions, in different contexts: Ambrus and Lu (2009) investigate multilateral bargaining with a deadline in a similar context, while Kamada and Kandori (2009) and Calcagno, Kamada, Lovo and Sugaya (2010) examine situations in which players can publicly modify their action plans before playing a normal-form game. An important difference between these models and ours, leading to different predictions, is that in the former models the actions players can take are unrestricted by previous history. In contrast, in our auction game, previous bids restrict the set of feasible bids thereafter, since the leading price can only increase. There is also a recent string of papers in industrial organization, on structural estimation of continuous-time models in which players can change their actions at discrete random times, but payoffs are accumulated continuously (Doraszelski and Judd (2011), Arcidiacono et al. (2013)).

2 Model and Benchmark Truthful Equilibrium

A continuous-time single-good auction is defined by a set of n potential bidders with reservation values $v_1, v_2, ..., v_n$ and arrival rates $\lambda_1, \lambda_2, ..., \lambda_n$, and a start time $T < 0$. We normalize the end time of the auction to 0. We assume that

al. (2012).

[7]Bounded arrival rates correspond to assuming that bidders cannot guarantee for sure that they arrive just before the end of the auction. In line with this point, 90 percent of bidders report, in Roth and Ockenfels's (2002) survey, that sometimes when they specifically plan to bid late, something comes up that prevents them from being available to bid at the end of the auction.

$v_i \in \mathbb{Z}_{++}$ and $\lambda_i \in \mathbb{R}_{++}$ for every $i = 1, ..., n$. For simplicity, for most of the paper we restrict attention to the case when $v_i = v$ for every $i = 1, ..., n$.[8] This setting corresponds to an environment in which the good has a known common value. While this is clearly a simplifying assumption, recent research on eBay suggests that a large number of auctions fall in this category. In particular, Einav et al. (2011) find hundreds of thousand cases on eBay in which the same seller simultaneously sells items with exactly the same description through auctions and also through a traditional posted price mechanism. The latter can be considered as the market price, or commonly known value of the particular good from the particular seller.

Between times T and 0 bidders get random opportunities to place bids according to independent Poisson processes with arrival rates as above. We normalize the starting bid to 0 and the minimum bid increment to 1. Bidders may make multiple bids during a single arrival and can observe the outcome of each bid. For simplicity we assume that all bids made during an arrival are carried out instantaneously.

We assume that bidders can leave proxy bids, hence we need to distinguish between current price p and current highest bid $B \geq p$. The set of available bids is given by $\{b \in \mathbb{Z}_{++} | b \geq B + 1\}$ for the bidder who holds the highest bid at time t and $\{b \in \mathbb{Z}_{++} | b \geq p + 1\}$ otherwise. When a bid b is made, the price adjusts as follows: if $b \geq B + 1$, then p becomes B, B becomes b, and the player who placed the bid becomes the winning bidder.[9] Otherwise, p becomes b and both B and the winning bidder remain the same. At the end of the auction ($t = 0$) the current high bidder wins the good and pays the current price. As in eBay auctions, we assume that the history of p and the identity of the winning bidder are publicly observed, but B is only known by the bidder holding the highest bid (provided that someone placed a bid).

The assumption that bidders can place multiple bids at an arrival opportunity enables them to bid incrementally, regardless of the current highest bid. In particular, a bidder can always bid the current price plus one, until she becomes the winning bidder. This is an important component of our model.

Strategies of bidders specify bidding behavior (that is either placing an available bid or not placing a bid) upon arrival as a function of calendar time, public history (time paths of p and the identity of the winning bidder) and private history (previous arrival times of the player and previous actions chosen at those arrival times). For expected payoffs to be well defined for all strategy profiles, we restrict bidders' strategies to be measurable with respect to the natural topologies.[10]

In the rest of the analysis we restrict attention to strategy profiles in which

[8] See Section 6 on asymmetric valuations, as well as more general environments with reservation values being private information.

[9] In a previous version of the model, we defined the rules such that p becomes $B + 1$, as opposed to B. This corresponds better to the Ebay mechanism, but makes the analysis more cumbersome. Nevertheless, the qualitative conclusions we obtained were the same in the two model versions.

[10] For the formal details, see Appendix A of Ambrus and Lu (2009) in a similar continuous-time game with random arrivals.

players' strategies only depend on the public history. The solution concept we use is perfect public equilibrium (subgame perfect Nash equilibrium in which all players use strategies that only depend on the public history) in conditionally weakly undominated strategies. From now on, for ease of exposition, we will simply refer to the concept as *equilibrium.* Note that without the requirement that strategies are not weakly dominated, even in sealed-bid second-price auctions typically there exist many equilibria, as low valuation bidders may place bids above their valuations, influencing the winning price, as long as they do not win the object.

In some of the analysis we further restrict attention to Markovian equilibria. We say that a bidder's strategy is Markovian if it only depends on payoff-relevant information, namely the current leading price p, whether the player is currently winning the object or not, and calendar time t. The latter is payoff relevant as it determines the distribution of future arrival sequences by the bidders (in particular, the probability that the given bidder will not get another chance to place a bid). In particular, a Markovian strategy depends trivially on the history of prices and winning bidders before t.

The weak undomination requirement, although considerably less restrictive in our game than in a static auction, is still relatively easy to check. In particular, it rules out placing bids above one's valuation after any history, and it does not rule out placing any bid at or below the true valuation, after any history. On top of this, the only additional restriction that conditional weak undomination imposes is that losing players cannot abstain from placing a bid close enough to the end of the auction if play at that point is consistent with the current highest bid being strictly below v.

In order to state this formally, for every $i = 1, ..., n$, let t_i^* be defined as the unique $t < 0$ satisfying $1 - e^{t\lambda_i} = e^{t\Sigma_{j\neq i}\lambda_j}$. Note that the left side is strictly decreasing in t, while the right side is strictly increasing. Moreover, for small enough t the left hand side is clearly larger than the right hand side, while for t close to 0 the right hand side is larger. Hence, t_i^* is well-defined. The interpretation of t_i^* is that it is the time at which bidder i is indifferent between becoming the winning bidder at some price $p < v$, but assuming that this event triggers every other bidder to place a bid of v upon first arrival, versus passing on the bidding opportunity and waiting for the next opportunity, assuming that if doing so, no other bidder will ever increase her bid. Intuitively, after t_i^* becoming the winning bidder is strictly preferred by i even when she has the most pessimistic belief regarding the continuation strategies of others conditional on this event, and the most optimistic belief regarding the continuation strategies conditional on not bidding at the current time.

Claim 1. A strategy of player i is conditionally weakly undominated iff it satisfies the following: (i) it never calls for placing a bid $b > v$ after any history; (ii) if at any time-t history such that $t \geq t_i^*$, player i is not the current winner, and given the history it is possible that she can become the winning bidder at a price $p < v$, it calls for placing a bid.

For the proof, see the Appendix. All of the equilibria we construct below trivially satisfy the conditions for conditionally weakly undominatedness.

It is easy to see that a strategy profile in which each bidder i bids v at any arrival when the current price is below v, and otherwise does not bid, constitutes an equilibrium. Given other bidders' strategies, no bidder can gain at any history by deviating from this strategy, and Claim 1 implies that these strategies are weakly undominated. Furthermore, there cannot be any equilibrium giving a higher expected revenue to the seller, given that in equilibrium as defined above, no player ever places a bid above her valuation. For this reason, and because it is analogous to the unique equilibrium in a second-price sealed bid auction, the above *truthful equilibrium* is a natural benchmark to compare all other equilibria to in the subsequent analysis.

3 Short Auctions

In this section we consider auctions that are short enough (relative to the arrival rates of bidders) that any losing bidder wants to place a bid when possible. For this case we analytically characterize a class of Markovian equilibria that includes the most gradual possible bidding equilibrium (that is when bidding always implies raising the price incrementally) on one extreme, and truthful equilibrium on the other. The former is the worst symmetric Markovian equilibrium for the seller in short auctions.

In Subsection 3.1 we provide an example of an incremental equilibrium, and explain the main features of the dynamic strategic interaction in this equilibrium. In Subsection 3.2 we formally characterize a class of Markovian equilibria with gradual bidding.

3.1 Example of a short auction with two bidders

Consider an auction with 2 symmetric bidders with values $v = 4$ and arrival rates $\lambda = 1$, and let $T = -1$. We would like to construct an equilibrium in which bidders make only the minimal increment necessary to become the wining bidder, whenever they arrive.

Formally, we consider a strategy profile in which a losing bidder bids $p + 1$ when $p \in \{0, 1, 2, 3\}$ and refrains from bidding when $p \geq 4$.[11] At the same time, a winning bidder refrains from increasing the current (proxy) price if she gets the chance to do so.

Let $W(p, t)$ and $L(p, t)$ denote the expected payoffs of a winning bidder (the bidder holding the current high bid) and the losing bidder respectively at time t when the current price is p, along the path of play when players adhere to the strategies above. Note that we suppress the current high bid as this is uniquely determined by the price along the path of play.

[11] Note that a losing bidder upon arrival bids up the price until he becomes the winning bidder upon arrival in this example. Thus a losing bidder on the equilibrium path will place multiple bids.

Trivially, $W(4,t) = 0$ and $L(p,t) = 0$ for $p \geq 3$. At $p = 2$ and $p = 3$ the winning bidder gets a payoff of $v - p$ if the other bidder does not get an arrival before the end of the auction and 0 otherwise. Therefore $W(3,t) = e^t$ and $W(2,t) = 2e^t$. The expected value of being a losing bidder at $L(2,t)$ is derived by using properties of a Poisson arrival process. Note that if there are at least two arrivals and players follow the bidding strategy described above, then the bidder obtains a payoff of zero. Thus the only event for which he obtains a positive payoff is if there is exactly one arrival of a losing bidder, in which case the payoff is $4 - 3 = 1$ since he must pay a price of 3. Given the Poisson arrival process, this event occurs with probability $-te^t$, which implies

$$L(2,t) = -te^t.$$

Similarly, we can compute $L(1,t) = -2te^t$. Following the same lines, the only events under which a winning bidder at a price of 1 obtains positive payoffs is if there are exactly zero or two arrivals, in which cases he obtains payoffs of 3 and 1, respectively, which implies

$$W(1,t) = 3e^t + \frac{t^2}{2}e^t.$$

Finally being the winning bidder at $p = 0$ gives the continuation payoff

$$W(0,t) = 4e^t + 2\frac{t^2}{2}e^t = 4e^t + t^2e^t$$

whereas being the losing bidder at $p = 0$ gives

$$L(0,t) = -3te^t - \frac{t^3}{3!}e^t.$$

Before any bids have been placed, neither bidder holds the high bid and so the expected payoff is the expectation over becoming either the winning bidder at $p = 0$ or the losing bidder at $p = 0$:

$$L(\emptyset,t) = \int_t^0 e^{-2(\tau-t)}(L(0,\tau) + W(0,\tau))d\tau.$$

Figure 1 depicts the expected continuation payoffs of winning and losing bidders at different prices, for the time horizon of the game. It is straightforward to check that for $t \geq -1$, all the incentive compatibility conditions hold for the above strategy profile to be an equilibrium. In particular, a losing bidder always prefers to take the lead upon an arrival, and being a winning bidder at a lower price is strictly better than at a higher price, providing an incentive for incremental bidding. In fact, at low prices bidders strictly prefer following the equilibrium strategy to any other action.

The intuition behind the above incremental bidding equilibrium is that if the opponent uses an incremental bidding strategy, a *losing bidder* faces a clear tradeoff in her bidding decision. On one hand, bidding makes her the current

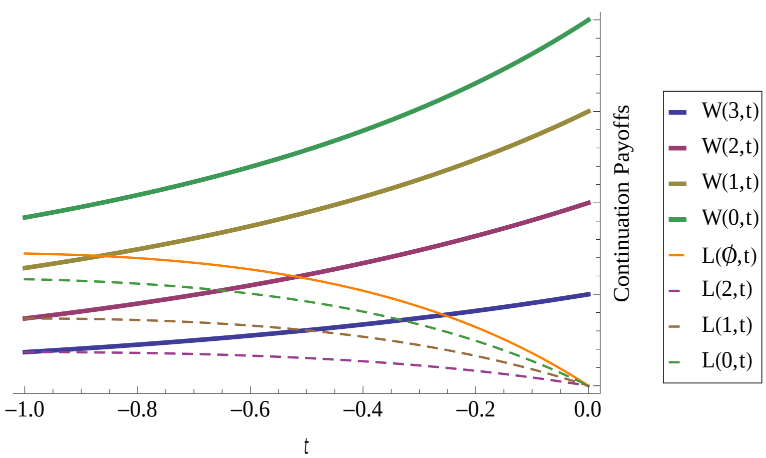

Figure 1: Continuation value functions for $T = -1$ and $v = 4$ in 2-bidder auction.

high bidder which increases her chance of winning the auction. The downside is that placing a bid activates the other bidder and raises the object's expected selling price. Placing a bid greater than the increment prescribed in equilibrium increases the downside without affecting the upside and hence if she chooses to place a bid it will also be incremental. Furthermore, the upside is increasing in t while the downside is decreasing in t. If an auction is short enough, it will support an incremental equilibrium in which bids are placed at every arrival by a losing bidder. This argument also hints that in longer auctions equilibrium also requires periods of waiting in which losing bidders pass on opportunities to bid, as the incentive to slow the increase of the current price might become stronger than the incentive to take the lead. We discuss incremental equilibria with delay in long auctions in Section 4.

Note that in the above equilibrium, a bidder's expected payoff is $L(\emptyset, -1) \approx 1.45$, and the seller's expected revenue is $(1 - e^{-2})4 - 2L(\emptyset, -1) \approx 0.57$. These expected payoffs are considerably more favorable to the bidders than those in the benchmark equilibrium, which are roughly 0.93 and 1.60 for the bidders and seller, respectively.[12]

[12] In short auction like this, different equilibria only affect the distribution of the surplus between the buyers and the seller, not total social welfare. In longer auctions, as demonstrated in the next section, there can be equilibria in which all bidders restrain from placing a bid until near the end of the auction, which does reduce social welfare relative to the benchmark truthful equilibrium.

3.2 Symmetric Markovian equilibria in short auctions

We now generalize the existence of equilibria with incremental bidding. In particular, we characterize a class of equilibria in which bidding behavior only depends on the current price and whether the bidder is currently winning.

Definition 1. A *bidding sequence* $S = \{b_1, ..., b_k\}$ is an integer-valued set that satisfies $0 < b_1 < ... < b_k$ and $b_k = v$. S is a *completely gradual bidding sequence* if $S = \{1, 2, \dots, v\}$.

Given a bidding sequence S and any price $p \leq v$, define

$$l_{p,S} = \min\{l : b_l \geq p\}.$$

If $p > v$, then define $l_{p,S} = v$. For the remainder of the paper, because the bidding sequence of interest will be unambiguous, we write l_p as shorthand for $l_{p,S}$. Then b_{l_p} is equal to p if $p \in S$ and otherwise it is the smallest element of S that is greater than p. With this we can formally introduce a class of Markovian strategies that we will study in this section.

Definition 2. A bidder bids *incrementally over bidding sequence* $S = \{b_1, ..., b_k\}$ *with no delays* by bidding

1. b_{l_p+2} if $l_p \leq k-2$ as a losing bidder,
2. v if $v > l_p > k-2$ as a losing bidder,
3. b_1 if no bids have been placed,
4. and otherwise refrains from bidding.

Furthermore a bidder bids *completely gradually with no delays* if S in the above definition is the completely gradual bidding sequence.

Note that if players follow the above strategy then the winning price at any moment is b_l for some l, the current highest bid is b_{l+1}, and the losing bidder upon an arrival plans to bid b_{l+2}. This relatively subtle way of prescribing strategies is necessary to induce players to place bids along a general bidding sequence, instead of deviating to bidding more gradually than what the sequence prescribes. Thus in checking incentive compatibility of the candidate strategy, it is sufficient to consider only deviations by bidders to bids in the bid sequence. As we saw in the example in the previous subsection, for the most gradual bidding sequence strategies can be defined in a simpler way: if the price is p, given the chance a losing bidder bids $p + 1$ (and if that was not enough to take the lead then she bids $p + 2$, etc.).

Theorem 1. *In an auction with n symmetric bidders, for any bidding sequence $S = \{b_1, \ldots, b_k\}$ with $b_k = v$, there exists a $t^* < 0$ such that the auction has an equilibrium in which bidders follow the incremental bidding strategy with no delays over S if and only if $T \geq t^*$. In particular, if there are two bidders, $t^* = -1/\lambda$.*

Proof. The following proof is for the two bidder case where we can show that incremental equilibria exist iff $T \geq -\frac{1}{\lambda}$. The proof for n bidders is conceptually the same but notationally more demanding, and it is given in the Appendix. We construct the expected continuation values recursively, with $L(b_k, t) = 0$ and $W(b_k, t) = (v - b_{k-1})e^{\lambda t}$ and for $0 < l < k$,

$$L(b_{k-l}, t) = \int_t^0 \lambda e^{-\lambda(\tau - t)} W(b_{k-l+1}, \tau) d\tau$$
$$W(b_{k-l}, t) = \int_t^0 \lambda e^{-\lambda(\tau - t)} L(b_{k-l+1}, \tau) d\tau + (v - b_{k-l})e^{\lambda t}$$

The following incentive compatibility conditions are sufficient to show that an incremental bidding strategy is a best response:

$$L(b_{k-l}, t) \geq L(b_{k-l+1}, t)$$
$$W(b_{k-l+1}, t) \geq L(b_{k-l}, t).$$

The first inequality ensures that incremental bids are weakly better than higher bids; higher bids weakly reduce the expected continuation value from becoming a losing bidder without affecting the expected continuation value from remaining the winning bidder until the end of the auction. Note that this also implies that winning bidders weakly prefer to not adjust their initial bid upon subsequent arrivals. The second inequality implies that incremental bidding is always weakly preferred to remaining a losing bidder. Note that the first inequality is always satisfied since for any realization of a sequence of arrivals, either both bidders in both scenarios become losers at the deadline obtaining a payoff of zero at the deadline or the players both obtain the final bid after which the losing bidder starting at a highest bid of b_{k-l} wins the auction at a weakly higher price than a losing bidder starting at a highest bid of b_{k-l+1}. We will now prove that the second inequality always holds with an inductive proof. Note the following expressions:

$$\begin{aligned} W(b_k, t) &= 0 \\ W(b_{k-1}, t) &= e^{\lambda t}(v - b_{k-1}) \\ L(b_{k-1}, t) &= 0 \\ L(b_{k-2}, t) &= -\lambda t e^{\lambda t}(v - b_{k-2}). \end{aligned}$$

Clearly $W(b_k, t) \geq L(b_{k-1}, t)$ and $W(b_{k-1}, t) \geq L(b_{k-2}, t)$ for all $t \geq -1/\lambda$. We now prove the inductive step; if $W(b_{k-l}, t) \geq L(b_{k-l-1}, t)$ for all $t \geq -1/\lambda$ then

$W(b_{k-l-2},t) \geq L(b_{k-l-3},t)$ for all $t \geq -1/\lambda$.

$$\begin{aligned}
L(b_{k-l-3},t) &= \int_t^0 \lambda e^{-\lambda(\tau-t)} W(b_{k-l-2},\tau)d\tau \\
&= -\lambda t e^{\lambda t}(v-b_{k-l-2}) + \int_t^0 \lambda e^{-\lambda(\tau-t)} \int_\tau^0 \lambda e^{-\lambda(s-\tau)} L(b_{k-l-1},s)dsd\tau \\
&\leq e^{\lambda t}(v-b_{k-l-2}) + \int_t^0 \lambda e^{-\lambda(\tau-t)} \int_\tau^0 \lambda e^{-\lambda(s-\tau)} W(b_{k-l},s)dsd\tau \\
&= W(b_{k-l-2},t),
\end{aligned}$$

where the inequality follows from our assumption that $W(b_{k-l},t) \geq L(b_{k-l-1},t)$ for all $t \geq -1/\lambda$. This proves that the second inequality holds for all l. At the beginning of the auction before any bid has been placed, both bidders are active. This expected continuation value is denoted by $L(\emptyset,t)$ and defined as:

$$L(\emptyset,t) = \int_t^0 \lambda e^{-2\lambda(\tau-t)}(W(0,\tau)+L(0,\tau))d\tau$$

Again we must prove that $W(0,t) \geq L(\emptyset,t)$ for $t \geq -1/\lambda$, since otherwise upon an arrival bidders would prefer to delay bidding. To prove this, we consider a modified game with price initialized at -1 and a modified bidding sequence for this game: $\tilde{S} = \{0, b_1, \ldots, b_K\}$. Let $\tilde{L}(p,t)$ and $\tilde{W}(p,t)$ be the corresponding continuation value functions in the modified game where $\tilde{L}(-1,t)$ is defined as:

$$\tilde{L}(-1,t) = \int_t^0 \lambda e^{-\lambda(\tau-t)} W(0,\tau)d\tau$$

and for any price $p \geq 0$, $\tilde{L}(p,t) = L(p,t)$ and $\tilde{W}(p,t) = W(p,t)$. Note that the inductive argument above implies that

$$\tilde{L}(-1,t) < \tilde{W}(0,t)$$

for all $t \geq -1/\lambda$ and thus $\tilde{L}(-1,t) < W(0,t)$ for all $t \geq -1/\lambda$. However $L(\emptyset,t) \leq \tilde{L}(-1,t)$ for all t since for every realization of arrivals from a highest bid of 0 at time t, the realized payoff of the losing player is weakly larger in the modified game than in the original game.[13] This allows us to conclude

$$L(\emptyset,t) < W(0,t)$$

for all $t \geq -1/\lambda$ as desired, which implies that it is always suboptimal for a losing bidder to pass on a bidding opportunity. The way strategies are constructed implies that neither underbidding nor overbidding (relative to the bid prescribed by the strategy profile) can be strictly profitable deviations. □

[13] Note that in the original game a highest bid of 0 at time t means that $p = 0$ and no player has bid yet, whereas in the modified game a highest bid of 0 at time t implies a price of $p = -1$.

The proof of Theorem 1 reveals that if $t > t^*$ then a losing bidder prefers taking the lead even when the other bidder's subsequent bid is sufficiently high to prevent the former bidder from obtaining any surplus from the auction. Intuitively, for short enough auctions there cannot be sufficient incentives to prevent losing bidders from overtaking the lead, since the probability that another bidder gets an opportunity to bid is low enough that even the most severe punishment by other bidders (switching to a truthful equilibrium) is insufficient to prevent such behavior.

Another observation is that, in a symmetric Markovian equilibrium, bidding cannot stop until the price reaches v, even though at $P = v - 1$ a bidder is indifferent between bidding of v and abstaining. This property holds because if players abstain from overbidding $P = v - 1$ with some positive probability, a bid of v becomes strictly better at any point than bidding $v - 1$ (the next lower weakly undominated bid). Hence, players would never bid $v - 1$. But then the same argument can be used iteratively to establish that players would never bid $v - 2$, $v - 3$ and so on, leading to the unraveling of any gradual bidding.[14]

The above two observations imply that in short enough auctions the worst symmetric Markovian equilibrium for the seller is given by the most gradual incremental bidding equilibrium - the one over $S = \{1, 2, ..., v\}$. For the worst equilibrium over all, which in short auctions has a very similar structure, see Subsection 4.2.

4 Long auctions

In this section we examine gradual bidding equilibria in auctions with longer time horizons. Maintaining such equilibria requires periods for which losing bidders abstain from bidding. In particular, players might wait to bid until relatively close to the end of the auction. Further periods of inactivity after a bid has already been placed, are also possible. Because of this, no matter how long the auction, the seller's expected revenue in these equilibria can be very small relative to v.

In Subsection 4.1, we provide an example of a gradual bidding equilibrium with waiting. In long auctions it becomes considerably easier to construct gradual bidding equilibria using non-Markovian strategies, so in Subsection 4.2 we propose a class of such equilibria, for any valuation and any number of bidders, that includes a completely gradual bidding equilibrium with bidding being delayed as much as possible. In Subsection 4.3 we focus on Markovian equilibria, and propose a simple gradual bidding equilibrium for any valuation and any number of bidders, and a completely gradual bidding equilibrium when the number of bidders is large enough.[15]

[14] There can be non-Markovian equilibria, as well as asymmetric Markovian equilibria in which bidding stops at $P = v - 1$.

[15] As our example in Subsection 4.1 shows, having a large number of bidders is not necessary for the existence of a completely gradual bidding Markovian equilibrium, but for a small number of bidders it becomes difficult to verify all incentive constraints for general valuations.

4.1 Example of a long auction with two bidders

The failure of the gradual bidding equilibria with no delays in long auctions can be seen by extending the length of the 2-bidder auction example from the previous section to $T = -2$. Figure 2 plots the non-trivial bidder value functions in the fully incremental equilibrium. As we demonstrated previously, for all p and $t > -1$, placing a bid is optimal as $W(p+1,t) > L(p,t)$. However, at any time $t < -1$, a winning bidder's expected value at $p = 3$ is lower than a losing bidder's expected value at $p = 2$ and hence the losing bidder facing a price of 2 would find it profitable in expectation to wait to bid until $t > -1$. Nonetheless, we can still construct equilibria with incremental bidding behavior in long auctions.

Sustaining incremental bidding in equilibrium requires intervals during which bidders abstain from bidding even though the price is below their value and they do not hold the current high bid. Bidders choose to wait when the cost of increasing the price outweighs the likelihood of winning the object with the current bid. In our example, the trade-off is straightforward. Since it induces the other player to bid again, bidding at $p = 2$ yields a positive payoff only in the event that the other bidder does not return to the auction. This event is less likely as we extend the time remaining in the auction. On the other hand, the likelihood of returning to the auction at the same price but closer to the end of the auction, and thereby face a more favorable trade-off, is increasing in the time remaining in the auction. For these reasons, early in the auction a losing bidder at $p = 2$ prefers waiting, while later he prefers taking the lead.

We refer to the point in time τ_p at which at which a bidder is indifferent between overtaking the current high bid at a current price p and waiting for the next opportunity, as the cutoff for price p. An incremental equilibrium with waiting is characterized by a bidding sequence and its corresponding sequence of cutoff points.

In an equilibrium in which bidders follow a symmetric Markovian incremental bidding strategy with delays, bidder value functions are constructed in the same manner as for incremental equilibria with no waiting. For example, when $S = \{1, 2, .., v\}$, the value functions are given by

$$L(p,t) = \int_{s_p^t}^{0} \lambda e^{-\lambda(\tau - s_p^t)} W(p+1,\tau)d\tau$$

$$W(p,t) = \int_{s_p^t}^{0} \lambda e^{-\lambda(\tau - s_p^t)} L(p+1,\tau)d\tau + (v-p)e^{\lambda s_p^t}$$

$$V(0,t) = \int_{s_{-1}^t}^{0} \lambda e^{-2\lambda(\tau - s_{-1}^t)} W(0,\tau)d\tau + \int_{s_{-1}^t}^{0} \lambda e^{-2\lambda(\tau - s_{-1}^t)} L(0,\tau)d\tau$$

where $s_p^t = \max\{t, \tau_p\}$ for all $p \geq 0$. Non-trivial cutoffs satisfy $L(p, \tau_p) = W(p+1, \tau_p)$ for all p and $V(0, \tau_{initial}) = W(0, \tau_{initial})$.

In our example, there are two relevant cutoffs (i.e. not equal to T); $\tau_0 = -\frac{17}{15}$ and $\tau_2 = -1$. Figure 3 plots the value functions for bidders following these

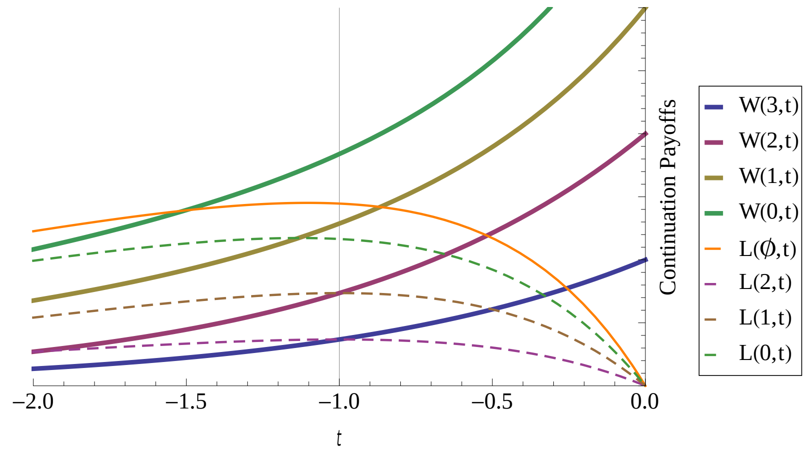

Figure 2: Continuation value functions with no delay with $v = 4$ in 2-bidder auction

cutoffs. The auction is divided into three periods; in the first period players initiate the bidding but further bidding does not take place, keeping the price at 0. In the second period, players bid incrementally if $p \in \{0, 1\}$, but pass on opportunities to bid if $p = 2$. Finally, in the third period a losing bidder bids until price reaches $p = 4$.

Bidders' equilibrium expected payoffs and the seller's expected revenue in this equilibrium are 1.44 and 0.58, respectively. The seller's expected revenue compares favorably to that of the short auction but it is still significantly less than in the benchmark equilibrium.

One feature of the above equilibrium is that the cutoff is non-trivial for every second price. The intuition behind this is as follows. Note that at any price, the winning bidder's expected value is greater than that of the losing bidder. Now suppose a losing bidder arrives at time t and faces a price $p-1$. If bidding at price p does not begin until $\tau_p > t$, a losing bidder cannot do better than to be the winning bidder at price p and at time τ_p; hence, the bidder must at least weakly prefer placing a bid. For this reason if there a nontrivial cutoff for overbidding p then the cutoff for overbidding $p-1$ has to be trivial.

4.2 Non-Markovian Equilibria in Long Auctions

As shown in Section 3, in short auctions it is relatively easy to construct gradual bidding equilibria for arbitrary bidding sequences, even in Markovian strategies. Verifying that the construction constitutes an equilibrium is facilitated by the fact that players trivially do not have an incentive to overbid (relative to what they are supposed to bid given the underlying bidding sequence). In equilibria with waiting, verifying that players have no incentives to overbid becomes non-

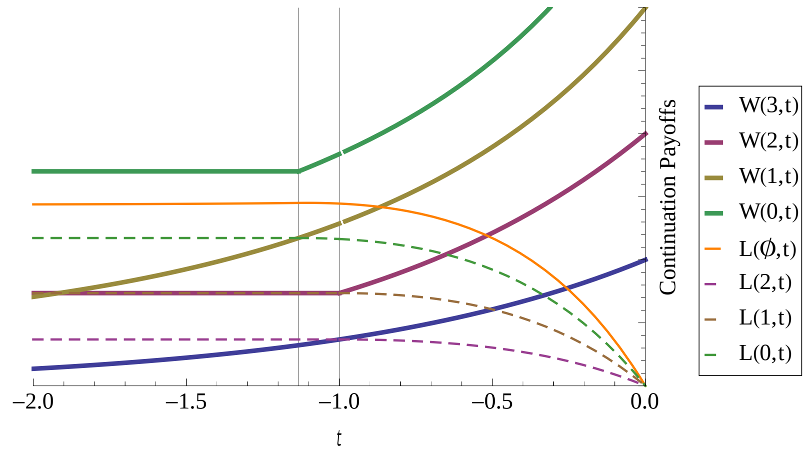

Figure 3: Equilibrium continuation value functions with delay with $v = 4$ in 2-bidder auction

trivial. To see this, consider a sequence of cutoffs $\{\tau_1, \tau_2, ..., \tau_{v-1}\}$ and a strategy profile according to which a losing bidder bids incrementally whenever the price is p and $t < \tau_p$, and otherwise passes on bidding opportunities. Suppose that τ_p for some p is a nontrivial cutoff. If a losing bidder i arrives at $t < \tau_p$ when price is $p-2$ then the above implies that the bidder faces a trade-off between placing a highest bid of p, as prescribed by the completely gradual bidding sequence, versus placing a bid of $p+1$. On one hand, the latter is better because it implies that if the other player gets a bidding opportunity between t and τ_p then she will bid p but refrain from further bidding. This ensures that at time τ_p bidder i remains the winning bidder. The downside of bidding $p+1$ versus p is that the former implies that if the next arrival by another player is after τ_p then she will not stop bidding at p, and takes over the highest bid anyway, but at a higher price than under gradual bidding.

In this subsection we avoid this complication by considering a class of non-Markovian equilibria in which players refrain from overbidding because the latter triggers a continuation equilibrium in which bidders switch to truthful strategies (placing a bid of v whenever possible), which is the most severe punishment possible in equilibrium. In particular, we focus on equilibria that yield the worst payoffs among equilibria in which bidding is along a particular bidding sequence S, by maximally delaying the period of refraining from placing a particular bid along the bidding sequence. We will consider Markovian gradual bidding equilibria in long auctions in the next subsection.

In order to define gradual bidding equilibria with periods of inactivity, we need to extend our definition of bidding sequences.

Definition 3. Let $S = \{b_1, \ldots, b_k\}$ be a bidding sequence. A strategy profile is an *incremental bidding strategy profile with delay over bidding sequence* S and cutoff sequence $C_S = \{t^{\emptyset}, t^0, \ldots, t^{k-2}\}$ if *on the equilibrium path*,

1. no bidding occurs when $t < t^{l_p+2}$,
2. losing bidders bid b_{l_p+2} if $t \geq t^{l_p+2}$ and $l_p \leq k-2$,
3. losing bidders bid v if $v > l_p > k-2$,
4. losing bidders bid b_1 if $p = \emptyset$ and $t \geq t^{\emptyset}$,
5. and otherwise bidders refrain from bidding.

Note that the above definition only characterizes bidding behavior on the equilibrium path. We leave the strategies off the equilibrium path of play unrestricted in the definition and show that there exist equilibria whose outcome path follows the definition above.

Theorem 2. *Let* $S = \{b_1, \ldots, b_k\}$*. Then there exists a cutoff sequence* C_S *such that there exists an equilibrium that is an incremental bidding strategy profile with delay over* S *and* C_S*. Moreover, among cutoff sequences which can constitute an equilibrium with the cutoff sequence, there is a maximal one* $\{t^{\emptyset}, t^0, \ldots, t^{k-2}\}$*, in the sense that* $t^i \geq \widehat{t}^i$ *for any* $i \in \{\emptyset, 0, 1, ..., k-2\}$ *and cutoff sequence* $\{\widehat{t}^{\emptyset}, \widehat{t}^0, \ldots, \widehat{t}^{k-2}\}$ *that can constitute an equilibrium with the same cutoff sequence.*

We will refer to the equilibrium in the second part of the statement as the maximally delayed equilibrium with the given bidding sequence. The construction of such an equilibrium is quite intuitive. We use reversion to the truthful equilibrium as punishment to deter deviations off the equilibrium path that involve overbidding relative to the equilibrium strategy.

Proof. We proceed in a recursive manner. Denote by $W^n(p,t)$ and $L^n(p,t)$ the value functions for the winning and losing bidders in an n player auction at price p and time t conditional on all players playing according to the incremental bidding strategy over S with no delay. Define t^l_n as the maximal time at which reversion to the truthful equilibrium is no longer sufficient to deter bidding at a price of b_l:

$$e^{\lambda(n-1)t^l_n}(v - b_{l+1}) = L^n(b_l, t^l_n).$$

Then define $t^{k-2} = t^{k-2}_n$ and also

$$W(b_{k-2}, t) = \begin{cases} W^n(b_{k-2}, t) & \text{if } t \geq t^{k-2} \\ W^n(b_{k-2}, t^{k-2}) & \text{if } t < t^{k-2} \end{cases}$$

$$L(b_{k-2}, t) = \begin{cases} L^n(b_{k-2}, t) & \text{if } t \geq t^{k-2} \\ L^n(b_{k-2}, t^{k-2}) & \text{if } t < t^{k-2}. \end{cases}$$

With this defined, we now define the other cutoffs recursively. Suppose that t^{l+1} and $W(b_{l+1},t)$ and $L(b_{l+1},t)$ have been defined. Then define the following value functions

$$\tilde{W}(b_l,t) = e^{\lambda(n-1)t}(v-b_l) + \int_t^0 \lambda e^{-\lambda(n-1)(\tau-t)}(n-1)L(b_{l+1},\tau)d\tau,$$

$$\tilde{L}(b_l,t) = \int_t^0 \lambda e^{-\lambda(n-1)(\tau-t)}\left(W(b_{l+1},\tau)+(n-2)L(b_{l+1},\tau)\right)d\tau.$$

These functions above are not intended to be the correct value functions $W(b_l,t)$. Rather $\tilde{W}(b_l,t)$ is defined to be the ex-ante continuation payoff associated a strategy in which the winner does not place any bids and a losing bidder bids b_{l+2} upon arrival leading to a continuation payoff of $L(b_{l+1},t)$. $\tilde{L}(b_l,t)$ is similarly defined. Thus these continuation values assume that any losing bidder places a bid upon arrival at any time after t. Using these value functions we can define the cutoff t^l as the time at which the threat of truthful bidding is no longer sufficient to deter bidding:

$$\tilde{L}(b_l,t^l) = e^{\lambda(n-1)t^l}(v-b_{l+1}).$$

Then we can define the true value functions as:

$$W(b_l,t) = \begin{cases} \tilde{W}(b_l,t) & \text{if } t \geq t^l \\ \tilde{W}(b_l,t^l) & \text{if } t < t^l, \end{cases}$$

$$L(b_l,t) = \begin{cases} \tilde{L}(b_l,t) & \text{if } t \geq t^l \\ \tilde{L}(b_l,t^l) & \text{if } t < t^l. \end{cases}$$

These definitions follow due to the fact that at any time t^l the threat of punishment is indeed effective in deterring lowing bidders from bidding. Thus the value functions at any time $t < t^l$ correspond to the value at time t^l. Iterating we can construct all of the relevant cutoffs $t^0, t^1, \ldots, t^{k-2}$ and all of the relevant continuation value functions. In a similar manner we construct the cutoff $t^{\emptyset}$. Given $W(0,t)$, $L(0,t)$, we define

$$\tilde{L}(\emptyset,t) = \int_t^0 \lambda e^{-\lambda n(\tau-t)}\left(W(0,\tau)+(n-1)L(0,\tau)\right)d\tau.$$

Then define $t^{\emptyset}$ as

$$\tilde{L}(\emptyset,t^{\emptyset}) = e^{\lambda(n-1)t^{\emptyset}}v.$$

Furthermore we define the continuation value when no player has bid as:

$$L(\emptyset,t) = \begin{cases} \tilde{L}(\emptyset,t) & \text{if } t \geq t^{\emptyset} \\ \tilde{L}(\emptyset,t^{\emptyset}) & \text{if } t < t^{\emptyset}. \end{cases}$$

With all of these cutoffs defined, we are now ready to define the candidate strategy profile. Define first the set $\mathcal{H}$:

$$\mathcal{H} = \{h \in H : (p,t) \in h \text{ and } p \notin S \text{ or } p > b_l, t < t^l \text{ for some } l\}.$$

In words this is the set of histories where either some player has been revealed to have bid some amount not in the bid sequence or to have placed a bid above b_l before time t^l. It turns out that these histories form the histories off the equilibrium path of play in the following candidate strategy profile. The candidate strategy profile is defined as follows.

1. If $p = \emptyset$, bid b_1 if and only if $t \geq t^{\emptyset}$.
2. If $h \notin \mathcal{H}$ and $\vec{h} = (p,t)$ with $p \neq \emptyset$, then bid b_{l_p+2} if and only if $t \geq t^{l_p}$ and the bidder is losing.
3. If $h \in \mathcal{H}$, bid v.
4. Otherwise refrain from bidding.

First note that any element of $h \in \mathcal{H}$ is not on the outcome path of play according to this strategy profile. With this observation, it is easy to check that each player has incentives to play according to the strategy specified above. □

The strategies constructed above have the feature that on the equilibrium path, bidding is incremental with delays according to the cutoff sequence $t^{\emptyset}, \ldots, t^{k-2}$. Such behavior is optimal due to the threat of reversion to the truthful equilibrium when players deviate by bidding when the strategy prescribes waiting.

For a general bidding sequence, the maximally delayed equilibrium constructed above can be complicated, with multiple subsequent waiting periods corresponding to different prices. In the Appendix we provide a sufficient condition on the bidding sequence for the maximally delayed equilibrium to have a simple structure, for any number of bidders, in which there are at most two effective cutoff times, and all waiting is frontloaded. There we also show that the same result holds for any bidding sequence when the number of bidders is large.

4.3 Markovian Equilibria in Long Auctions

Constructing Markovian equilibria for general bidding sequences in long auctions is complicated, for the reasons spelled out at the beginning of Subsection 4.2. However, in this subsection we show that for any number of bidders and any valuation there exists a Markovian equilibrium with some gradual bidding. Even for this equilibrium the seller's revenue is low in general, as in a long auction players are inactive for most of the auction. We also show that if the number of bidders is large enough, a Markovian equilibrium exists in which bidding is completely gradual. In fact, this result can be generalized to any bidding sequence.

4.3.1 A Simple Gradual Bidding Markovian Equilibrium

First we show that for any time horizon, any number of bidders, and any value for the object, there exists a Markovian equilibrium with gradual bidding. In particular, we show that the bidding sequence $\{1, v\}$ with the cutoff sequence $\{T, -1/\lambda\}$, constitutes a Markovian equilibrium. Note that Markovian strategies pin down play off the equilibrium path as well, hence the latter do not need to be specified separately.

Claim 2. The following symmetric strategy profile constitutes a Markovian equilibrium. If no one has bid, a player upon an arrival bids 1. If someone is winning the object at a price of 0, a losing bidder upon an arrival refrains from bidding for $t < -1/\lambda$, and bids v for $t \geq -1/\lambda$. If someone is winning the object at a price $1 \leq p \leq v-1$, a losing bidder upon an arrival bids v. If someone is winning the object at a price $p \geq v$, a losing bidder upon an arrival refrains from bidding. Lastly, a winning bidder always refrains from further bidding.

Proof. The expected continuation payoff of a bidder winning the object at price 1 at time t, assuming that other bidders play the prescribed profile is $W(1,t) = (v-1)e^{t\lambda(n-1)}$. The expected continuation payoff of a losing bidder at some time earlier than t, when all losing bidders including him refrain from bidding until time t is $\int_t^0 \lambda e^{-(n-1)\lambda(s-t)} W(1,s)ds = -t\lambda e^{t\lambda(n-1)}(v-1)$. This expression is smaller than $W(1,t)$ exactly for $t < -1/\lambda$. Hence, if other bidders follow the prescribed strategies, it is indeed optimal for a losing bidder at price 0 to refrain from overbidding until $t = -1/\lambda$, and placing a bid afterwards. All other incentive constrains trivially hold in the candidate equilibrium profile. □

We now calculate the seller's expected profits when $T \leq -1/\lambda$ under this equilibrium and show that they can be small:

$$\begin{aligned}
&e^{\lambda nt} \cdot 0 + (-\lambda nt)e^{\lambda nt} \cdot 0 \\
&+ \frac{(-\lambda nt)^2}{2} e^{\lambda nt} \cdot 1 + \left(1 - e^{\lambda nt} - (-\lambda nt)e^{\lambda nt} - \frac{(-\lambda nt)^2}{2} e^{\lambda nt}\right) v \\
&= \frac{(-\lambda nt)^2}{2} e^{\lambda nt} + \left(1 - e^{\lambda nt} - (-\lambda nt)e^{\lambda nt} - \frac{(-\lambda nt)^2}{2} e^{\lambda nt}\right) v,
\end{aligned}$$

where $t = -1/\lambda$. Simplifying the above expression gives:

$$\frac{n^2}{2} e^{-n} + \left(1 - e^{-n} - ne^{-n} - \frac{n^2}{2} e^{-n}\right) v.$$

Not surprisingly, the seller's expected profits converge to v as $n \to \infty$. However for small n, the seller's profits can be relatively small even if v is large. For example when $n = 3$ and $v = 10$, the seller's profit is 5.995 and when $n = 4$ and $v = 10$, the seller's profit is 7.77. However in this equilibrium, the seller's profits converge to v very fast as n increases. Thus profits are close to v in such equilibria unless the number of bidders participating in the auction is very small.

4.3.2 Completely Gradual Bidding in Markovian Equilibrium

Here we examine the existence of equilibria that exhibit completely gradual bidding, that is when along the equilibrium path every time the winning bidder changes, the winning price only increases by the minimum bid increment.

Definition 4. A strategy profile is a *Markovian completely gradual bidding strategy profile with delays over the cutoff sequence* $C = \{t^{\emptyset}, t^0, \ldots, t^{v-2}, t^{v-1}\}$ if the bidder bids $p+1$ at a price of p and time t if and only if she is a losing bidder and $t \geq t^p$. Otherwise she refrains from bidding.

Note first that these strategies restrict play at histories on and off the equilibrium path, which differentiates these strategies from those of the previous section where behavior off the equilibrium path was left flexible. For this reason, the equilibrium constructions of Subsection 4.2 exhibit more delay in bidding because players can use the harshest punishment available, namely reversion to the truthful equilibrium, to deter any deviations. In this section, such use of punishment is prohibited as the strategies studied here are more restrictive. The next result states that for the completely gradual bidding sequence, if the number of bidders is large enough, there exists an equilibrium that is a Markovian completely gradual bidding strategy profile with delays over some cutoff sequence.

Theorem 3. *There exists an n^* such that for all $n > n^*$, there exists a cutoff sequence $C = \{t^{\emptyset}, t^0, \ldots, t^{v-2}, t^{v-1}\}$ such that the strategy profile in which all players bid completely gradually with delays over C is an equilibrium.*

To prove this result, we first let the cutoffs be the earliest times at which a losing bidder at a given price p would prefer becoming the winning bidder at price $p+1$ rather than waiting, conditional on all bidders in the future following a strategy profile of complete gradual bidding with no delays. These cutoffs can be defined uniquely for each n. Moreover, for large n, we show that this cutoff sequence is monotonic so that

$$t^{v-1} < t^{v-2} < \cdots < t^0 < t^{\emptyset}.$$

It is now easy to define continuation values consistent with completely gradual bidding over this cutoff sequence. Because the cutoff sequence is strictly decreasing, bidders in the proposed equilibrium refrain from bidding until $t^{\emptyset}$, after which they engage in completely gradual bidding, with no delays. Because of this, one can check the incentive compatibility of the strategies following a technique similar to the one used in the section on short auctions.

In the Appendix we prove a more general version of this result, showing that for any bidding sequence, when the number of bidders is large enough then there exists a cutoff sequence that together with the bidding sequence constitutes a Markovian equilibrium.

5 Comparative Statics

In this section, we investigate how the seller's expected profits depend on the basic parameters of the game in the seller's worst case scenario: the maximally delayed non-Markovian equilibria corresponding to the completely gradual bidding sequence. This exercise provides bounds on how large the seller's losses can be in gradual bidding equilibria relative to the truthful equilibrium when the value of the object or the number of bidders is taken to infinity.

In Appendix A.4 we show that the maximally delayed equilibrium corresponding to the completely gradual bidding sequence has the property that, for any n and v, there are at most two effective cutoffs, t^0 and $t^{\emptyset}$. That is, all players abstain from bidding until $t^{\emptyset}$, and then depending on the relative magnitudes of the above cutoffs either immediately engage in completely gradual bidding, or abstain from overbidding an initial bid until t^0, and then engage in completely gradual bidding afterwards. Since t^0 and $t^{\emptyset}$, for fixed n and v, are independent of T, the above implies that comparative statics are trivial with respect to T. In particular, the seller's revenue weakly increases in $|T|$, but $|T| > \max(|t^{\emptyset}|, |t^0|)$ implies that further increases in $|T|$ do not affect the seller's revenue. Hence for long enough auctions the seller's expected revenue is (locally) independent of T, and strictly less than v.

Below we examine how the seller's revenue depends on v and n, assuming that the length of the auction is long enough such that $|T| > \max(|t^{\emptyset}|, |t^0|)$. As a first step, we explicitly calculate the cutoffs t^0 and $t^{\emptyset}$. Note that

$$e^{\lambda(n-1)t^0}(v-1) = L(0, t^0).$$

In Appendix A.1, we show that the above holds if and only if

$$v - 1 = \sum_{j=0}^{v-1} \frac{(-\lambda t^0)^j}{j!} \left(\frac{(n-1)^j + (-1)^{j+1}}{n}(v-j) \right).$$

Because the right side is 0 when $t^0 = 0$ and strictly increases toward ∞ as $t \to -\infty$, the equality must have a unique solution.

To calculate the cutoff $t^{\emptyset}$, first define $\hat{t}$ by:

$$v = \sum_{j=0}^{k-1} (-1)^{j+1} \left(\sum_{l=j+1}^{\infty} \frac{\left(\lambda \hat{t}\right)^l}{l!} \right) (n-1)^j (v - b_j).$$

Again using the expressions derived in Appendix A.1, note that $\hat{t}$ is the time at which $ve^{\lambda(n-1)\hat{t}}$ equals the bidder's continuation value at price $\emptyset$ when all players follow a completely gradual bidding strategy with no future delays.

We can now determine $t^{\emptyset}$. If $\hat{t} \geq t^0$, then $t^{\emptyset} = \hat{t}$. If $\hat{t} < t^0$, $t^{\emptyset}$ is defined by explicitly recalculating the value function $L(\emptyset, t)$ taking into account that players delay at a price of $p = 0$ before time t^0. Note that for all $t \geq t^0$,

$$L(\emptyset, t) = e^{\lambda(n-1)t} \sum_{j=0}^{k-1} (-1)^{j+1} \left(\sum_{l=j+1}^{\infty} \frac{(\lambda t)^l}{l!} \right) (n-1)^j (v - b_j)$$

However for $t < t^0$, we have

$$L(\emptyset, t) = \left(1 - e^{-\lambda n(t^0 - t)}\right)\left(\frac{1}{n}W(0, t^0) + \frac{n-1}{n}L(0, t^0)\right) + e^{-\lambda n(t^0 - t)}L(\emptyset, t^0).$$

Therefore $t^{\emptyset}$ uniquely satisfies

$$\left(1 - e^{-\lambda n(t^0 - t^{\emptyset})}\right)\left(\frac{1}{n}W(0, t^0) + \frac{n-1}{n}L(0, t^0)\right) + e^{-\lambda n(t^0 - t^{\emptyset})}L(\emptyset, t^0) = e^{\lambda(n-1)t^{\emptyset}}v.$$

5.1 Changes in the Object's Value

Changes in v have two opposing effects on the seller's revenue. An increase in v pushes $t^{\emptyset}$ (as well as t^0) closer to the deadline. Thus there are more delays in equilibrium, raising the possibility that few bids are placed. However a higher v means bidders are more willing to bid up the price if, during the active bidding period of the auction, they receive enough bidding opportunities. Below we show that the first effect always dominates and thus the seller's expected revenue is always decreasing in v.

To study the effects of changing v on the seller's revenue analytically, in Appendix B.1, we investigate how the two potential effective cutoffs change when v increases, and find the following: (i) t_v^0 increases as v increases, and it converges to a limit strictly below 0 as $v \to \infty$; (ii) similarly $t_v^{\emptyset}$ increases as v increases and converges to a limit strictly below 0 as $v \to \infty$.

However we show that the increases in $t^{\emptyset}$ must be big relative to the benefits to the seller of an increase in v, thus causing expected revenue to decrease. Intuitively, the benefits from an increase in the object's value from v to $v + 1$ are only realized if there are at least $v + 1$ bidding opportunities during the active bidding period. But since the active bidding period is relatively small, the probability of the price reaching $v + 1$ is small. In contrast, bidders benefit from an increase in the object's value even after just one bidding opportunity during the active bidding period. With this intuition, one can argue that the seller's expected revenue must decrease with v. However, since $t^{\emptyset}$ and t^0 converge to limits strictly below zero as $v \to \infty$, profits cannot vanish since there is a strictly positive probability of at least two bidders arriving even when v is very large. Collectively, these observations yield the following:

Claim 3. For the maximally delayed equilibrium corresponding to the completely gradual bidding sequence, given a fixed n, the seller's expected revenue is decreasing in v and converges to some $\pi^* > 0$ as $v \to \infty$.

5.2 Changes in the Number of Bidders

Similar to increases in the object's value, there are two opposing effects on the seller's revenue when the number of bidders increases. The more direct effect is that for fixed cutoff points, the expected number of bids and hence the winning price is higher. However, when n increases, there is a more subtle negative impact on the seller's revenue, because cutoffs become closer to the deadline. Analyzing the trade-off between these two effects analytically is complicated in general, but below we show that profits converge to v in the limit as $n \to \infty$. Hence, for large number of bidders the first effect dominates.

To show the above, consider the auxiliary strategy (not necessarily an equilibrium) where at some time $\hat{t}_n$, a bidder i is chosen to be a winner at price -1 according to a uniform random draw. No players bid at any time $t < \hat{t}_n$. All players except player i follow a completely gradual bidding strategy with no delays after $\hat{t}_n$. Bidder i only bids after another player has bid following $\hat{t}_n$, after which he follows completely gradual bidding with no delays.

Choose $\hat{t}_n$ to satisfy the following:

$$ve^{\lambda(n-1)\hat{t}_n} = \frac{1}{n} e^{\lambda(n-1)\hat{t}_n} \sum_{\ell=0}^{v} \frac{(-\lambda(n-1)\hat{t}_n)^\ell}{\ell!}(v - \ell + 1).$$

Note that such a $\hat{t}_n$ exists when n is sufficiently large. The right side represents the ex-ante expected continuation value at time $\hat{t}_n$ to following the above strategy, where $\hat{t}_n > t_n^{\emptyset}$ since the continuation value at $\hat{t}_n$ must be at least the value to playing the completely gradual bidding strategy.

But note that the above implies

$$nv = \sum_{\ell=0}^{v} \frac{(-\lambda(n-1)\hat{t}_n)^\ell}{\ell!}(v - \ell + 1).$$

The left side converges to ∞ as $n \to \infty$. Thus we must have $\lambda(n-1)\hat{t}_n \to -\infty$ as $n \to \infty$, which implies that $\lambda(n-1)t_n^{\emptyset} \to -\infty$ and $\lambda n t_n^{\emptyset} \to -\infty$.

Consequently, expected profits converge to v as $n \to \infty$. Thus when n is sufficiently large, the effect of an increase in the cutoff time $t_n^{\emptyset}$ is dominated by the benefit of having more bidders, allowing the seller to extract essentially all of the consumer's surplus.

However, numerical estimations show that the convergence is rather slow, especially for more valuable objects. As the table below shows, for $v = 10$, even when the number of bidders is 15, the seller's expected revenue is only a small fraction of the object's value.

n	2	3	5	10	15
π	0.549	0.844	1.275	1.965	2.407

Table 1: $v = 5$

n	2	3	5	10	15				
π	0.504	0.758	1.127	1.715	2.100				

Table 2: $v = 10$

6 Extensions

In this section we show that the existence of equilibria with gradual bidding and waiting generalizes to the case of bidders with asymmetric valuations, as well as to situations in which bidders are uncertain about the valuations of other bidders. As these environments are analytically more difficult, we restrict our attention to the case of two possible valuations. We also discuss how the results extend when allowing for time-dependent arrival rates.[16]

6.1 Asymmetric Values

Here we consider the case with two bidders, who have commonly known but different valuations for the object. We show that for any pair of valuations with the feature that even the lower valuation exceeds the minimum bid, there exists an equilibrium in which the initial bid by a low valuation bidder is incremental. In this equilibrium, the low valuation bidder wins the object with non-trivial frequency for auctions of arbitrary length. In the Appendix we also provide an example in which both bidders bid incrementally, and discuss the generalization of this example.

Proposition 1. *For any 2-bidder auction with bidder values $v_H > v_L \geq 2$, symmetric arrival rates and $|T|$ sufficiently large, there exists an equilibrium with gradual bidding.*

The equilibrium we construct is such that the low valuation bidder, when not winning the object, always places a bid upon arrival, as long as current price is below v_L. In particular, she bids 1 if no one placed a bid before, and places a bid of v_L if the high valuation bidder is the current winner. The high valuation bidder's strategy is characterized by two cutoff points, $t_\emptyset^H$ and t_0^H. She abstains from bidding before $t_\emptyset^H$ if no one has bid beforehand and also before $t_0^H > t_\emptyset^H$ if the low valuation bidder is the current winner, but bids v_H otherwise. Hence, in this profile the low valuation bidder bids gradually, while the high valuation bidder waits until near the end of the auction to bid. The high valuation bidder waits because, bidding too early increases the probability that the ultimate winning price is v_L instead of 1.

There are a couple instructive features of the equilibrium. First, in the Appendix we show that the indifference conditions for the two cutoffs of the

[16] In a previous version of the paper we also showed that the types of gradual bidding equilibria we construct also exist when bidders have heterogenous arrival rates.

high evaluation bidder are:

$$v_H - 1 + \lambda t_0^H (v_L - 1) = 0$$
$$v_H - 1 + \lambda t_\emptyset^H (v_L - 1) = e^{\lambda t_\emptyset^H}.$$

Given this, for a fixed v_H, increasing v_L increases both t_0^H and $t_\emptyset^H$. The high valuation bidder is induced to wait to reduce the probability that the low valuation bidder bids again to raise the price to v_L. As v_L becomes large relative to the certain payoff from bidding truthfully, the incentive to wait strengthens and hence the high valuation bidder is willing to wait longer.

Second, even when the auction is arbitrarily long, the low valuation bidder wins the auction with nontrivial probability, and achieves a substantial payoff. To see this, consider an auction with $v_H = 6$, $v_L = 4$, $T = -\infty$, and $\lambda = 1$. In the benchmark truthful equilibrium, the high and low valuation bidders respectively bid 6 and 4 at their first opportunity. This implies that with probability 1, the high valuation bidder wins and gets a payoff of 2 giving the seller a payoff of 4. In contrast, in the equilibrium constructed in Claim 3, the low valuation bidder's expected payoff is $v_L e^{\lambda t_0^H} = 4e^{-\frac{5}{3}} \approx 0.76$. Because the low valuation bidder can only win at price $p = 0$, the low type has approximately a 19% chance of winning the auction whereas the high valuation bidder has a 81% chance. The total expected payoff among both bidders is equal to 3.32 (with a payoff of 2.57 to the high type) versus 2 in the benchmark equilibrium, and the seller's expected revenue falls to roughly 2.3. Since the losing bidder places at least one bid with probability one, there is no inefficiency in this equilibrium due to no bidding. There is however inefficiency due to the fact that the low valuation bidder wins the object with some probability.

6.2 Asymmetric Information

Thus far we have considered auctions in which each bidder's value is common knowledge. We now consider the case when valuations are privately known. For simplicity, we restrict attention to two bidders with identically and independently drawn valuations with binary support. In this environment it is possible to construct equilibria in which a bidder can only make inferences on the other bidder's type once it is no longer relevant to her bidding decisions. This greatly simplifies the calculation of cutoff points and incentive constraints. However, we conjecture that the games at hand have many more complicated incremental equilibria in which bidders draw nontrivial inferences on each others' types along the equilibrium path.

Proposition 2. *Assume that there are n bidders, whose valuations are drawn iid, taking value $v_L > 1$ with probability $q \in (0, 1)$ and $v_H > v_L$ with probability $1 - q$. Then there exists an equilibrium in which bidders high types bid gradually, and both types of bidders abstain from winning at certain histories along the equilibrium path.*

The equilibrium we construct in the proof of Proposition 2 is such that the first bidder with an opportunity to bid bids v_L (irrespective of her type), after which players abstain from bidding until a cutoff time t^*. After t^*, bidders bid truthfully. The cutoff point for jump bidding is decreasing in q, the likelihood that a bidder is a low type. A high type risks less by outbidding early as the likelihood that her opponents are low types increases. When $q = 0$ and $t^* = -1/\lambda$, the game reduces to the symmetric complete information case. For long auctions, if there are two bidders, each bidder has a likelihood of $\frac{1}{2}$ of being the winning bidder at t^* and the likelihood that the other bidder gets no bidding opportunities after t^* is $e^{\lambda t^*}$; hence, a low type bidder playing against a high type bidder wins the auction with positive payoff with approximate likelihood of $0.5e^{\lambda t^*}$. For low values of q with $\lambda = 1$, this likelihood is quite high; at $q = .1$, a low type playing against a high type wins with probability 0.165.

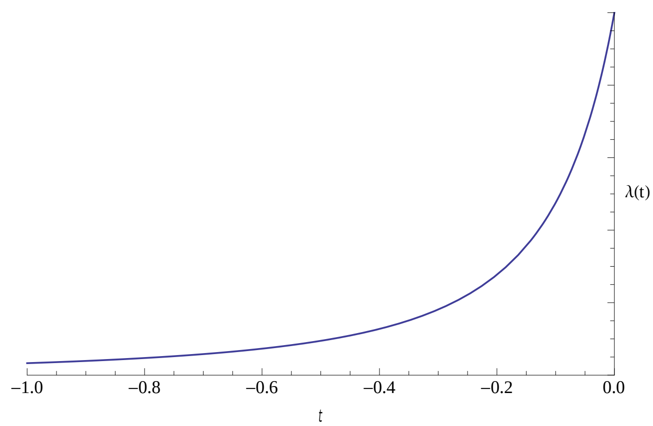

Figure 4: A time-dependent arrival rate defined by $\lambda(t) = \frac{a}{(1-bt)^2}$ with $a = 10, b = \frac{9}{2}$

6.3 Time-dependent arrival rates

First, we note that multiplying all arrival rates by a constant $\alpha > 0$ is equivalent to rescaling time by $\frac{1}{\alpha}$. In particular, if the original game has an incremental bidding strategy equilibrium over bidding sequence $\{b_1, ..., b_k\}$ and cutoff sequence $\{t_1, ..., t_k\}$ then the game where arrival rates are multiplied by α has an incremental bidding strategy equilibrium over bidding sequence $\{b_1, ..., b_k\}$ and cutoff sequence $\{\frac{1}{\alpha}t_1, ..., \frac{1}{\alpha}t_k\}$. Furthermore, expected payoffs with time horizon T in the original game are the same as with time horizon $\frac{T}{\alpha}$ in the game with the rescaled arrival rates. In particular, if $T \leq t_1$ then increasing arrival rates while keeping T fixed does not change the expected equilibrium payoffs: it only shifts all cutoffs closer to the deadline. Intuitively, if bidders get frequent bidding opportunities, it makes them postpone bidding at different prices, in a way that exactly offsets the effect of increasing the arrival rates.

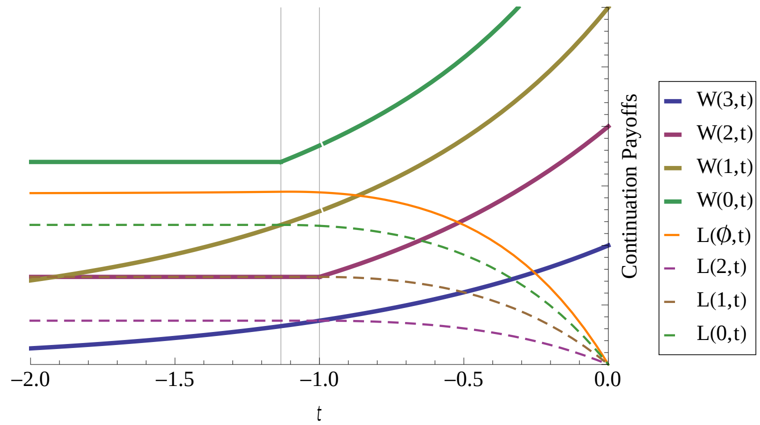

Figure 5: Equilibrium continuation values with $\lambda = 1$

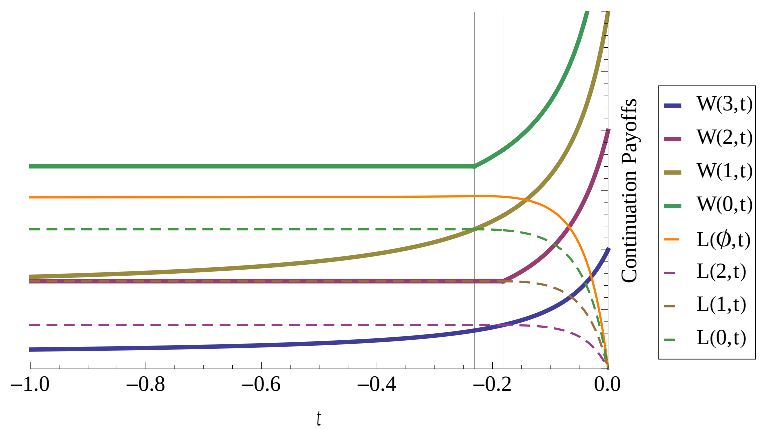

Figure 6: Equilibrium continuation values with time-dependent λ

We return to the symmetric 2-bidder auction example studied in Section 4.1 where $v = 4$ and $T = -2$ but now let the arrival rate be a strictly increasing function $\lambda(t) = \frac{a}{(1-bt)^2}$ where $a, b > 0$. The arrival rate at the end of the auction, $\lambda(0)$, is then equal to a, while b determines how steeply arrival rates increase at the end of the auction. The average arrival rate over the auction is given by $\bar{\lambda} = \int_{-2}^{0} \frac{a}{(1-bt)^2} dt$. We choose $a = 10$ and $b = 9/2$ illustrated in Figure 4, which gives an average arrival rate of 1 as in our original example. Figure 6 shows the effect of an increasing arrival rate on bidder value functions in the most gradual equilibrium. The structure of the equilibrium is identical: in the case with fixed arrival rates bidding begins after an initial cutoff and bidders wait to bid at $p = 1$ until after a second cutoff. However, with increasing arrival rates, the cutoffs are closer to the end of the auction, which reinforces the high frequency of late-bidding in equilibrium. As long as arrival rates are bounded, incremental equilibria are robust to increasing arrival rates.

7 Conclusion

This paper shows that in online auctions like eBay where bidders can leave proxy bids, if bidders get random chances to bid then many different equilibria arise in weakly undominated strategies. Bidders can implicitly collude by bidding gradually or by waiting to bid, in a self-enforcing manner, slowing the increase of the leading price. These features of our model are consistent with the empirical observations that both gradual bidding and sniping are common bidder behaviors on eBay.

Our investigation suggests that given a fixed set of bidders, running an ascending auction with a long time horizon (long enough that bidders cannot continuously participate) has the potential to adversely affect the seller's revenue, even when proxy bidding is possible, relative to running a prompt auction. Hence, introducing a time element can only be beneficial if it takes time for potential bidders to find out about the auction.[17] It is an open question what mechanism guarantees the highest possible revenue for the seller in such environments. In order to prevent implicit collusive equilibria, sellers might want to set high reservation prices and/or minimum bid increments. They might also want to allow each bidder to submit at most one bid over the course of an auction, although in practice this might be difficult to enforce, given that the same person can have multiple online identities. We leave the formal investigation of these issues to future research.

[17] In a recent paper, Fuchs and Skrzypacz (2010) consider the arrival of new buyers over time, but in a dynamic bargaining context in which the seller cannot commit to a mechanism. Another difference compared to our setting is that in their model once a buyer arrives, she is continuously present until the end of negotiations.

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A Appendix

A.1 Preliminaries for Equilibrium Analysis

In this section, we will show that the value functions for an n-bidder auction along the bid sequence $S = \{b_1, \ldots, b_k\}$ with no delays take the following form:[18]

$$W(b_{k-l}, t) = e^{\lambda(n-1)t} \sum_{j=0}^{l-1} \frac{(-\lambda t)^j}{j!} \left(\frac{(n-1)^j}{n} + \frac{(-1)^j (n-1)}{n} \right) (v - b_{k-l+j}) \quad (1)$$

$$L(b_{k-l}, t) = e^{\lambda(n-1)t} \sum_{j=0}^{l-1} \frac{(-\lambda t)^j}{j!} \left(\frac{(n-1)^j}{n} + \frac{(-1)^{j+1}}{n} \right) (v - b_{k-l+j}) \quad (2)$$

$$L(\emptyset, t) = e^{\lambda(n-1)t} \sum_{j=0}^{k-1} (-1)^{j+1} \left(\sum_{l=j+1}^{\infty} \frac{(\lambda t)^l}{l!} \right) (n-1)^j (v - b_j). \quad (3)$$

These expressions are quite easily obtained when $n = 2$. This is quite easy due to the fact that, given gradual bidding with no delays over S, the outcome path of winning bidders is simply an alternation between the two players. However, when $n > 2$, there are many more possible permutations of bidding outcomes on the equilibrium path. Thus the derivation of the above expressions is more complicated.

These closed form expressions of the value functions will be useful in our analysis of long auctions, as they remain valid in those auction for times close enough to the deadline. In particular, for a large number of players they enable us to show monotonicity of cutoffs in prices, which greatly simplify verifying that the strategy profiles we consider constitute equilibria.

A.1.1 A Simple Markov Chain

To derive the expressions above, we first analyze a simple Markov chain whose state space is the set of players $\{1, \ldots, n\}$. The transition matrix for this finite state Markov chain is as follows:

$$M \equiv \begin{pmatrix} 0 & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ \frac{1}{n-1} & 0 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & 0 \end{pmatrix}.$$

The interpretation for this Markov chain is the following. The state variable $s \in \{1, \ldots, n\}$ represents the identity of the winner at each stage. The process then transitions to a new winner $s' \in \{1, \ldots, n\} - s$ in the next period according to the above transition matrix. According to the transition matrix, each of the bidders who is not a winner has equal probability of becoming a winner in the

[18] Here we assume the convention that $0^0 = 1$ and that $0! = 1$.

next stage. Note that this transition matrix is important for our analysis, since the compound Poisson arrival process with jumps represented by this transition matrix represents the equilibrium outcome path of strategies in gradual bidding with no delays.

We first analyze the matrix M^j which represents the conditional probabilities of each individual being a winner at the end of j arrivals. Let E be a matrix whose elements consist of only ones. Then

$$
\begin{aligned}
M^j &= \frac{1}{(n-1)^j}(E-I)^j \\
&= \frac{1}{(n-1)^j}\sum_{l=0}^{j}\left(\begin{array}{c} j \\ l \end{array}\right)(-1)^l E^{j-l} \\
&= (-1)^j\frac{1}{(n-1)^j}I + \frac{1}{(n-1)^j}\sum_{l=0}^{j-1}\left(\begin{array}{c} j \\ l \end{array}\right)(-1)^l n^{j-l-1}E \\
&= (-1)^j\frac{1}{(n-1)^j}I + \frac{1}{(n-1)^j}\frac{E}{n}\sum_{l=0}^{j-1}\left(\begin{array}{c} j \\ l \end{array}\right)(if-1)^l n^{j-l} \\
&= \frac{1}{(n-1)^j}\left((-1)^j I + \frac{((n-1)^j + (-1)^{j+1})}{n}E\right).
\end{aligned}
$$

Then if a bidder is a winner, the probability that he is a winner at the end of j arrivals is equal to $d_j = M^j_{1,1} = M^j_{2,2} = \cdots = M^j_{n,n}$ which is

$$
d_j = \frac{1}{n} + \frac{(-1)^j}{n(n-1)^{j-1}}.
$$

Similarly if he is a loser, the probability that he becomes a winner at the end of j arrivals is equal to

$$
f_j = \frac{1}{n} + \frac{(-1)^{j+1}}{n(n-1)^j}
$$

which is equal to the elements of M^j not on the diagonal.

A.1.2 Value functions

We now use the derivations above to arrive at the following formula for the value functions of a winning and losing bidder at price b_{k-l} conditional on all players following an incremental bidding strategy with no delays over the bid sequence S.

$$
\begin{aligned}
W(b_{k-l}, t) &= e^{\lambda(n-1)t} \sum_{j=0}^{l-1} \frac{(-\lambda t)^j}{j!} (n-1)^j d_j (v - b_{k-l+j}) \\
&= e^{\lambda(n-1)t} \sum_{j=0}^{l-1} \frac{(-\lambda t)^j}{j!} \left(\frac{(n-1)^j + (-1)^j (n-1)}{n} \right) (v - b_{k-l+j}) \\
L(b_{k-l}, t) &= e^{\lambda(n-1)t} \sum_{j=0}^{l-1} \frac{(-\lambda t)^j}{j!} (n-1)^j f_j (v - b_{k-l+j}) \\
&= e^{\lambda(n-1)t} \sum_{j=0}^{l-1} \frac{(-\lambda t)^j}{j!} \left(\frac{(n-1)^j + (-1)^{j+1}}{n} \right) (v - b_{k-l+j})
\end{aligned}
$$

It remains to compute the value function $L(\emptyset, t)$.

To do this, we calculate $W(0, t)$ and $L(0, t)$ using the above expressions. Setting $k = l$, the above implies

$$
\begin{aligned}
W(0, t) &= e^{\lambda(n-1)t} \sum_{j=0}^{k-1} \frac{(-\lambda t)^j}{j!} \left(\frac{(n-1)^j + (-1)^j (n-1)}{n} \right) (v - b_j) \\
L(0, t) &= e^{\lambda(n-1)t} \sum_{j=0}^{k-1} \frac{(-\lambda t)^j}{j!} \left(\frac{(n-1)^j + (-1)^{j+1}}{n} \right) (v - b_j).
\end{aligned}
$$

Then we can calculate $L(\emptyset, t)$:

$$
\begin{aligned}
L(\emptyset, t) &= \int_t^0 \lambda e^{-\lambda n(\tau - t)} (W(0, \tau) + (n-1) L(0, \tau)) d\tau \\
&= e^{\lambda n t} \int_t^0 \lambda e^{-\lambda \tau} e^{-\lambda(n-1)\tau} (W(0, \tau) + (n-1) L(0, \tau)) d\tau \\
&= e^{\lambda n t} \int_t^0 \lambda e^{-\lambda \tau} \left(\sum_{j=0}^{k-1} \frac{(-\lambda \tau)^j}{j!} (n-1)^j (v - b_j) \right) d\tau \\
&= e^{\lambda(n-1)t} \sum_{j=0}^{k-1} (n-1)^j (v - b_j) e^{\lambda t} \int_t^0 \lambda e^{-\lambda \tau} \frac{(-\lambda \tau)^j}{j!} d\tau.
\end{aligned}
$$

We can simplify the above expression further with the following lemma.

Lemma 1.

$$h_j(t) \equiv e^{\lambda t} \int\limits_t^0 \lambda e^{-\lambda\tau} \frac{(-\lambda\tau)^j}{j!} d\tau = (-1)^j \left(\left(\sum_{l=0}^{j} \frac{(\lambda t)^l}{l!} \right) - e^{\lambda t} \right)$$
$$= (-1)^{j+1} \sum_{l=j+1}^{\infty} \frac{(\lambda t)^l}{l!}$$

Proof. The proof is shown by induction on j and integration by parts. The claim is obvious for $j = 0$. Now suppose that the claim holds for j. Then

$$\begin{aligned}\int\limits_t^0 \lambda e^{-\lambda\tau} \frac{(-\lambda\tau)^{j+1}}{(j+1)!} d\tau &= \frac{(-\lambda t)^{j+1}}{(j+1)!} e^{-\lambda t} - \int\limits_t^0 \frac{(-\lambda\tau)^j}{j!} \lambda e^{-\lambda\tau} d\tau \\ &= \frac{(-\lambda t)^{j+1}}{(j+1)!} e^{-\lambda t} + (-1)^{j+1} \left(e^{-\lambda t} \left(\sum_{l=0}^{j} \frac{(\lambda t)^l}{l!} \right) - 1 \right) \\ &= (-1)^{j+1} \left(e^{-\lambda t} \left(\sum_{l=0}^{j+1} \frac{(\lambda t)^l}{l!} \right) - 1 \right).\end{aligned}$$

This concludes the proof. ∎

This then gives us the expressions for the value functions given by equations (1), (2), and (3) at the beginning of this section.

A.1.3 Properties of the Value Functions

In this section, we prove some basic properties of the value functions derived in the previous section. These properties will be used in the subsequent proofs.

Lemma 2. *$L^n(b_{k-l}, t) > L^n(b_{k-l+1}, t)$ for all t and all $l \geq 2$. Also, $W^n(b_{k-l}, t) > W^n(b_{k-l+1}, t)$ for all t.*

The proof is obvious from the expressions in (1) and (2), hence omitted. Next we use this lemma to prove some basic properties of the value functions.

Lemma 3. *The following hold:*

1. *$e^{-\lambda(n-1)t} L^n(b_{k-l}, t)$ is strictly decreasing for all $t < 0$ for all $l \geq 2$.*
2. *$\frac{\partial}{\partial t} W^n(b_{k-l}, t) > 0$ when $W^n(b_{k-l}, t) \geq L^n(b_{k-l-1}, t)$.*
3. *$\frac{\partial}{\partial t}(W^n(b_{k-l+1}, t) - L^n(b_{k-l}, t)) > 0$ when $W^n(b_{k-l+1}, t) \geq L^n(b_{k-l}, t)$.*
4. *$e^{-\lambda(n-1)t} L^n(\emptyset, t)$ is strictly decreasing for all $t < 0$.*

5. $\frac{\partial}{\partial t}(W^n(0,t) - L^n(\emptyset,t)) > 0$ *when* $W^n(0,t) \geq L^n(\emptyset,t)$.

Proof. Consider the first statement.

$$e^{-\lambda(n-1)t} L^n(b_{k-l},t) = \sum_{j=0}^{l-1} \frac{(-\lambda(n-1)t)^j}{j!} f_j(v - b_{k-l+j})$$

The derivative of the right-hand side with respect to t is:

$$-\lambda(n-1) \sum_{j=1}^{l-1} \frac{(-\lambda(n-1)t)^{j-1}}{(j-1)!} f_j(v - b_{k-l+j}) < 0.$$

Now consider the second statement.

$$W^n(b_{k-l+1},t) = e^{\lambda(n-1)t}(v - b_{k-l+1}) + \int_t^0 \lambda e^{-\lambda(n-1)(\tau-t)}(n-1)L^n(b_{k-l+2},\tau)d\tau.$$

Using the fundamental theorem of calculus, and rearranging, the derivative of the right-hand side is:

$$\lambda(n-1)(W^n(b_{k-l+1},t) - L^n(b_{k-l+2},t)).$$

Note that the expression above is positive since

$$W^n(b_{k-l+1},t) \geq L^n(b_{k-l},t) > L^n(b_{k-l+2},t).$$

We establish statement 3 with a similar argument.

$$L^n(b_{k-l},t) = \int_t^0 \lambda e^{-\lambda(n-1)(\tau-t)}(W^n(b_{k-l+1},\tau) + (n-2)L^n(b_{k-l+1},\tau))d\tau.$$

Again using the fundamental theorem of calculus, the derivative of the right-hand side is

$$\lambda((n-1)L^n(b_{k-l},t) - W^n(b_{k-l+1},t) - (n-2)L^n(b_{k-l+1},t)).$$

Therefore

$$\begin{aligned}\frac{\partial}{\partial t}(W^n(b_{k-l+1},t) - L^n(b_{k-l},t)) &= \lambda(n-1)(W^n(b_{k-l+1},t) - L^n(b_{k-l},t)) \\ &\quad + \lambda(W^n(b_{k-l+1},t) - L^n(b_{k-l+2},t)) \\ &\quad + \lambda(n-2)(L^n(b_{k-l+1},t) - L^n(b_{k-l+2},t)).\end{aligned}$$

All of the terms above are positive, proving statement 3. Statement 4 is proved in the same manner as claim 1.

$$e^{-\lambda(n-1)t} L^n(\emptyset,t) = \sum_{j=0}^{k-1} h_j(t)(n-1)^j(v - b_j)$$

Note that

$$\begin{aligned}\frac{\partial}{\partial t}h_j(t) &= \lambda\left(h_j(t) - \frac{(-\lambda t)^j}{j!}\right)\\ &= \lambda\left((-1)^j\left(\left(\sum_{l=0}^{j}\frac{(\lambda t)^l}{l!}\right) - e^{\lambda t}\right) - \frac{(-\lambda t)^j}{j!}\right)\\ &= \lambda\left((-1)^{j+1}\left(\sum_{l=j}^{\infty}\frac{(\lambda t)^l}{l!}\right)\right) < 0\end{aligned}$$

for all $t < 0$. This then implies that $e^{-\lambda(n-1)t}L^n(\emptyset, t)$ is strictly decreasing in t for all $t < 0$.

$$\frac{\partial}{\partial t}W^n(0,t) = \lambda(n-1)(W^n(0,t) - L^n(b_1,t)).$$

Now consider the derivative of the value function $L^n(\emptyset, t)$.

$$\frac{\partial}{\partial t}L^n(\emptyset,t) = \lambda n L^n(\emptyset,t) - \lambda W^n(0,t) - \lambda(n-1)L^n(0,t).$$

Thus

$$\begin{aligned}\frac{\partial}{\partial t}(W^n(0,t) - L^n(\emptyset,t)) &= \lambda n(W^n(0,t) - L^n(\emptyset,t))\\ &\quad + \lambda(n-1)(L^n(0,t) - L^n(b_1,t)).\end{aligned}$$

Note that all of the terms in the above expression are positive. This proves statement 5. ∎

An immediate corollary of lemma above is the following.

Corollary 1. ***(Single-Crossing Property)*** *Suppose $W^n(b_{k-l+1}, t^*) = L^n(b_{k-l}, t^*)$. Then $W^n(b_{k-l+1}, t) > L^n(b_{k-l}, t)$ for all $t > t^*$ and $W^n(b_{k-l+1}, t) < L^n(b_{k-l}, t)$ for all $t < t^*$. Similarly suppose $W^n(0, t^*) = L^n(\emptyset, t^*)$. Then $W^n(0,t) > L^n(\emptyset, t)$ for all $t > t^*$ and $W^n(0,t) < L^n(\emptyset,t)$ for all $t < t^*$.*

Proof. Suppose $W^n(b_{k-l+1}, t^*) = L^n(b_{k-1}, t^*)$. Then by the previous lemma, $\frac{\partial}{\partial t}(W^n(b_{k-l+1}, t^*) - L^n(b_{k-l}, t^*)) > 0$. This immediately implies that $W^n(b_{k-l+1}, t) > L^n(b_{k-l}, t)$ for all $t > t^*$. This proves the first part of the claim. Now suppose that for some $t' < t^*$, $W^n(b_{k-l+1}, t') \geq L^n(b_{k-l}, t')$. Because $\frac{\partial}{\partial t}(W^n(b_{k-l+1}, t^*) - L^n(b_{k-l}, t^*)) > 0$, there exists some $\epsilon > 0$ such that $W^n(b_{k-l+1}, t) < L^n(b_{k-l}, t)$ for all $t \in (t^* - \epsilon, t^*)$. But then this implies that there exists some $t'' \in [t', t^*)$ such that $W^n(b_{k-l+1}, t'') = L^n(b_{k-l}, t'')$. From what we already proved above, this means that $W^n(b_{k-l+1}, t) > L^n(b_{k-l}, t)$ for all $t > t''$, which is a contradiction. The claim for the value functions $W^n(0,t)$ and $L^n(\emptyset, t)$ can be proved in exactly the same manner. ∎

A.2 Proof of Claim 1

Proof of Claim 1: A strategy that at a given history h calls for placing a bid of $b > v$ is conditionally weakly dominated by a strategy that at h calls for placing a bid of v if $v > P$ and abstaining from bidding otherwise, and specifies the same behavioral strategy as the original strategy at any other history. Take now any history h satisfying the requirements in part (ii) of the statement (player i is a losing bidder at h, but $B < v$ is consistent with h). For any possible $B < v$ given h, any continuation strategy that specifies abstaining from bidding at time t gives at most $(v - B)(1 - e^{t\lambda_i})$ expected payoff, while a continuation strategy that calls for incrementally bidding until either becoming the winning bidder or price reaching v yields at least $(v - B)e^{t\Sigma_{j\neq i}\lambda_j}$. Since $t > t_i^*$, the latter expected payoff is strictly larger. For any possible $B \geq v$ given the history, a continuation strategy that calls for incrementally bidding until either becoming the winning bidder or price reaching v yields a payoff of 0, which is the best payoff the player can get given that $B \geq v$, and hence yields at least weakly larger payoff than any continuation strategy that specifies not placing a bid at h. This concludes that any strategy that specifies not placing a bid at h is conditionally weakly dominated. The above conclude that (i) and (ii) are necessary for a strategy to be conditionally weakly undominated. For the remainder of the proof, if a strategy of any player i prescribes placing a bid at an arrival event at some time t for which i is already the winning bidder, by the prescribed action we mean the ultimate bid that i places at t following this history (independently of whether i places lower bids first at t before placing the ultimate bid). This is only to simplify exposition in the rest of the proof. Let now $t \leq t_i^*$, and h be a time t history at which player i has the opportunity to take an action. Consider any strategy s_i satisfying (i) and (ii), and assume that there exists another strategy s_i' weakly dominating s_i, conditional on h. This implies that there exists a time $t' \geq t$ history h' that is a successor of h (in a weak sense, that is it can be h itself) at which s_i and s_i' specify different actions and s_i' weakly dominates s_i, conditional on h'. In particular, there exists a strategy profile s_{-i} of the other players consistent with h' such that if other players play s_{-i} then conditional on h' s_i' yields a strictly higher payoff than s_i. Since s_i satisfies condition (i), i's expected payoff conditional on h' is nonnegative when playing s_i, therefore it has to be strictly positive when playing s_i' (given that the strategy played by the others is s_{-i}). This in turn implies that if the others play s_{-i} then at h' it has to be that $P < v$, and either i is the winning bidder or $B < v$. Next, in order to derive a contradiction, we will show that the above imply that there is a strategy profile s_{-i}' consistent with h' for which s_i yields a strictly higher payoff, conditional on h', than s_i'. Since at h' strategies s_i and s_i' specify different actions, we can have the following possibilities: Possibility 1: s_i specifies not placing a bid and and s_i' specifies placing a bid of b'. Consider first the case when i is the winning bidder at h'. Note that it cannot be that $B \geq v - 1$ at h', since then for any strategy of the others, s_i and s_i' induce the same path of play until some player other than i places a bid of v or larger. But in this case player i's payoff becomes 0 when playing s_i, and it becomes at most 0 when playing s_i. The above imply that s_i' cannot weakly dominate s_i, conditional on h'. Let now $B < v-1$ at h'. In this case let $s_{-i}' = s_{-i}$ for all information sets at all times preceding t', and for all information sets at times larger than t' let it prescribe the following strategy for all players other than i: bid $v-1$ if $P < v-1$, and immediately after that bid v if the former bid did not take over the lead; otherwise do not place any bid. Given this strategy of the others, player i's expected payoff conditional on h' is $(v-P)e^{t'\sum_{j\neq i}\lambda_j}$ (where P is the winning price at h') when playing s_i', and strictly higher than that when playing s_i. Next consider the case when $j \neq i$ is the winning bidder at h'. Note that it has to be consistent with h' that $B < v$, otherwise the best expected payoff conditional on h' that i can get is 0, which is guaranteed by s_i, which therefore could not be weakly dominated conditional on h'. Also note that $t' < t^*$, otherwise since s_i satisfies condition (ii) of the claim, it would have to specify placing a bid at h'. Given this, for all information sets at all times preceding t', let $s_{-i}' = s_{-i}$, with the exception that whenever s_{-i} specifies placing a bid at least P, s_{-i}' specifies placing a bid of $v-1$. Note that this implies $B = v-1$ at h'. For any information set at all times after t', let s_{-i}' specify the following strategies for all the other players: if along the history leading to the information set the winning bid changed at t', bid v whenever $P < v$; otherwise do not place a bid. Given this strategy of the others, player i's expected payoff conditional on h' is $(v-1)e^{t'\sum_{j\neq i}\lambda_j}$ when playing s_i', while it is at most $(v-1)(1-e^{t^*\lambda_i})$ when playing s_i, given that s_i satisfies condition (ii) of the claim. By the definition of t^*, the latter is strictly higher for $t' < t^*$. Possibility 2: s_i specifies placing a bid of b and and s_i' specifies not placing a bid. First, note that it cannot be that at h' the winning bidder is i, and $B = v-1$ (which, since s_i satisfies condition (i), implies $b = v$). This is because then for any strategy of the others, s_i and s_i' induce the same continuation play until one of the other players bid v or more. But in the latter case player i's payoff becomes 0 when playing s_i, and it becomes at most 0 when playing s_i'. This contradicts that s_i' weakly dominates s_i conditional on h'. Assume now that at h' the winning bidder is i, and $B < v-1$. In this case let $s_{-i}' = s_{-i}$ for all information sets at all times preceding t', and for times larger than t' let it prescribe the following strategy for all players other than i: place a bid of $B+1$ whenever $P < B+1$, otherwise abstain from bidding. If the others play s_{-i}', player i's expected payoff conditional on h' is $(v-B)e^{t'\sum_{j\neq i}\lambda_j} + (v-B-1)(1-e^{t'\sum_{j\neq i}\lambda_j})$ when playing s_i, and strictly less than that when playing s_i'. Consider now the case when at h' the winning bidder is $j \neq i$. Let P be the winning price at h'. In this case, for all information sets at all times preceding t', let $s_{-i}' = s_{-i}$, with the exception that whenever s_{-i} specifies placing a bid larger than P, s_{-i}' specifies placing a bid of exactly P. Note that this implies $B = P$ at h'. For all information sets at all times after t', s_{-i}' specifies not placing any bid. Then the expected payoff of i conditional on h' is exactly $v-P$ when playing s_i, and strictly less than that when playing s_i'. Possibility 3: both s_i and s_i' specify placing bids, but s_i

specifies placing a higher bid. Let b be the bid specified by s_i, and b' be the bid specified by s_i'. Note that it cannot be that $b' \geq v-1$, since then for any strategies of the others s_i and s_i' induce the same continuation play until one of the other players bid v or more. But in this case player i's payoff becomes 0 when playing s_i, and it becomes at most 0 when playing s_i'. This contradicts that s_i' weakly dominates s_i conditional on h'. Given this, for all information sets at all times preceding t', let $s_{-i}' = s_{-i}$, with the exception that whenever s_{-i} specifies placing a bid larger than P (the winning price at h'), s_{-i}' specifies placing a bid of exactly P. For all information sets at all times after t', s_{-i}' specifies placing a bid of $b'+1$ if $P < b'+1$, and not placing a bid otherwise. For this strategy of the others, conditional on h' the expected payoff of i is $(v-P)e^{t'\sum_{j\neq i}\lambda_j} + (v-b'-1)(1-e^{t'\sum_{j\neq i}\lambda_j})$ when playing s_i, and strictly less than that when playing s_i'. Possibility 4: both s_i and s_i' specify placing bids, but s_i specifies placing a lower bid, labeled b. First, consider the case when the winning bidder at h' is i. Notice that in this case it cannot be that $b \geq v-1$ since then for any strategies of the others s_i and s_i' induce the same continuation play until one of the other players bid v or more. But in this case player i's payoff becomes 0 when playing s_i, and it becomes at most 0 when playing s_i'. This contradicts that s_i' weakly dominates s_i conditional on h'. Given this, let $s_{-i}' = s_{-i}$ for all information sets at all times preceding t', and for times larger than t' let it prescribe the following strategy for all players other than i: place a bid of $b+1$ when $P < b+1$ and the winning bidder is i, and immediately after this place a bid of v if the previous bid did not take over the lead; do not place a bid at any other information set. Note that given this strategy of others, the expected payoff of player i conditional on h' is $(v-P)e^{t'\sum_{j\neq i}\lambda_j}$ when playing s_i', and strictly higher than that when playing s_i, given that s_i satisfies property (ii). Next, consider the case when the winning bidder at h' is $j \neq i$. Let b' be the bid specified by s_i' at h'. If $b' > v$, for all information sets at all times preceding t', let $s_{-i}' = s_{-i}$, with the exception that whenever s_{-i} specifies placing a bid of at least P, s_{-i}' specifies placing a bid of $v+1$. This implies that player i's expected payoff conditional on h' is strictly negative when playing s_{-i}', contradicting that the latter weakly dominates s_{-i} conditional on h'. Consider now $b' \leq v$. Note that this implies $b \leq v-1$. In this case, for all information sets at all times preceding t', let $s_{-i}' = s_{-i}$, with the exception that whenever s_{-i} specifies placing a bid of at least P, s_{-i}' specifies placing a bid of $v-1$. Not that this implies $B = v-1$ at h'. For any information set after t', let s_{-i}' specify the following actions: if along the history leading to this information set at time t' someone placed a bid of b but did not take over the lead (in particular the winning price went up to b) then do not place any bids; in any other case place a bid of v whenever $P < v$. Given this strategy of the others, if $t' \geq t^*$ then player i's expected payoff conditional on h' is exactly 1 when playing s_i (to see this, note that since s_i satisfies condition (ii) of the claim, it has to specify keeping on bidding at h' until i is winning at price $v-1$), while it is $e^{t'\sum_{j\neq i}\lambda_j} < 1$

when playing s'_i. If $t' < t^*$ then player i's expected payoff conditional on h' is at least $1 - e^{t^* \lambda_i}$ when playing s_i, since s_i satisfies condition (ii) of the claim, while it is $e^{t' \sum_{j \neq i} \lambda_j}$ when when playing s'_i. By the definition of t^*, the latter is strictly smaller. This concludes that there is no strategy weakly dominating s_i conditional on h'. □

A.3 Proof of Theorem 1 (n-bidders)

Given a strategy with gradual bidding over the sequence $S = \{b_0, \ldots, b_k\}$, we know from the previous section that the continuation values are given by the following expressions:

$$W(b_{k-l}, t) = e^{\lambda(n-1)t} \sum_{j=0}^{l-1} \frac{(-\lambda t)^j}{j!} \left(\frac{(n-1)^j}{n} + \frac{(-1)^j (n-1)}{n} \right) (v - b_{k-l+j})$$

$$L(b_{k-l}, t) = e^{\lambda(n-1)t} \sum_{j=0}^{l-1} \frac{(-\lambda t)^j}{j!} \left(\frac{(n-1)^j}{n} + \frac{(-1)^{j+1}}{n} \right) (v - b_{k-l+j})$$

$$L(\emptyset, t) = e^{\lambda(n-1)t} \sum_{j=0}^{k-1} (-1)^{j+1} \left(\sum_{l=j+1}^{\infty} \frac{(\lambda t)^l}{l!} \right) (n-1)^j (v - b_j).$$

Then we can define t^* as the infimum over all times t at which the following hold:

$$W(b_{k-l+1}, t) \geq L(b_{k-l}, t) \text{ for all } l = 1, \ldots, k, \text{ and}$$
$$W(b_0, t) \geq L(\emptyset, t).$$

Note that $t^* > -\infty$, since for every l, $W(b_{k-l+1}, t) \to 0$ as $t \to -\infty$ and at all times t at which $W(b_{k-l+1}, t) > L(b_{k-l}, t)$, $L(b_{k-l}, t)$ is decreasing in t. Thus the above shows that $W(b_{k-l+1}, t)$ must eventually cross $L(b_{k-l}, t)$, which allows us to conclude that $t^* > -\infty$.

If $T < t^*$, it is clear that no equilibrium involving gradual bidding with no delays exists as some losing bidders would prefer to not place a bid upon arrival at some times $t < t^*$.

If $T \geq t^*$, then incentives to underbid hold by construction of the strategies: On the equilibrium path, the belief of any losing bidder is that the highest bid is b_{k-l+1} whenever the price is b_{k-l}. Therefore he has no incentive to bid anything lower than b_{k-l+1} because he would remain the losing bidder. Thus it is a best response for him to simply bid b_{k-l+2} immediately. At histories off of the equilibrium path, the losing bidders in a perfect Bayesian equilibrium can hold any beliefs about the highest bid. In particular, we assume that players believe that the highest bid is b_{k-l_p+1}. This then makes a bid of b_{k-l_p+2} a best response for any losing bidder. Finally we need to check incentives to overbid. This is due to the monotonicity of the payoff functions in the price:

$$L(b_{k-l}, t) > L(b_{k-l+1}, t)$$

for all $l = 1, \ldots, k$. This can be checked by simply inspecting the expressions for the value function of a winning bidder computed above. However the monotonicity of the value functions of the losing bidder simply means that placing a bid higher than the specified bid only reduces payoffs. This is because either there are no more bids in which case, the payoffs of the two strategies will be the same. If there is another bid, then the player becomes a losing bidder at a higher price. Because this happens with positive probability, it cannot be a best response for the losing bidder to overbid. $\Box$

A.4 Non-Markovian Equilibria with Delays: Additional Results

Let $S = \{b_1, \ldots, b_k\}$ be a bidding sequence. We call a bidding sequence regular if the following assumption holds:

$$\frac{v - b_{k-l}}{v - b_{k-l-1}} \leq \frac{v - b_{k-l-1}}{v - b_{k-l-2}}.$$

A sufficient condition for the above to hold is if the increments are weakly decreasing and thus for example, the completely gradual bidding sequence satisfies the above property.

Theorem 4. *Let $S = \{b_1, \ldots, b_k\}$ be a regular bidding sequence. Then the cutoff sequence belonging to a maximally delayed equilibrium satisfies that t^l is decreasing in l for $l \in \{0, 1, ..., k-2\}$.*

Note that the statement implies that there can only be either one or two effective cutoffs along the equilibrium path: either $t^{\emptyset}$ and t^0, or only $t^{\emptyset}$. This is because players only overbid a winning price of 0 after t^0, which is weakly later than all cutoffs belonging to higher prices, hence bidding from this time on is gradual with no waiting, just like in a short auction.

Proof: The key step in the proof is the following lemma.

Lemma 4. *If $S = \{b_1, \ldots, b_k\}$ is regular, the continuation values for the losing bidders conditional on all bidders following the incremental bidding sequence with no delays over S have the following property:*

$$\frac{L^n(b_{k-l}, t)}{L^n(b_{k-l-1}, t)} \leq \frac{v - b_{k-l+1}}{v - b_{k-l}}.$$

Proof. First note the following continuation values:

$$\begin{aligned} L^n(b_{k-1}, t) &= 0 \\ L^n(b_{k-2}, t) &= -\lambda t e^{\lambda(n-1)t}(v - b_{k-1}) \\ L^n(b_{k-3}, t) &= -\lambda t e^{\lambda(n-1)t}(v - b_{k-2}) + \frac{\lambda^2 t^2}{2} e^{\lambda(n-1)t}(n-2)(v - b_{k-1}). \end{aligned}$$

Then note that

$$\begin{aligned}\frac{L^n(b_{k-1},t)}{L^n(b_{k-2},t)} &\leq \frac{v-b_k}{v-b_{k-1}},\\ \frac{L^n(b_{k-2},t)}{L^n(b_{k-3},t)} &\leq \frac{v-b_{k-1}}{v-b_{k-2}}.\end{aligned}$$

Now suppose that

$$\begin{aligned}\frac{L^n(b_{k-l+1},t)}{L^n(b_{k-l},t)} &\leq \frac{v-b_{k-l+2}}{v-b_{k-l+1}},\\ \frac{L^n(b_{k-l+2},t)}{L^n(b_{k-l+1},t)} &\leq \frac{v-b_{k-l+3}}{v-b_{k-l+2}}.\end{aligned}$$

Then we can write the value function for the losing bidder as:

$$\begin{aligned}\frac{L^n(b_{k-l},t)}{v-b_{k-l+1}} &= -\lambda t e^{\lambda t(n-1)} + \int_t^0 \lambda(n-2)e^{-\lambda(\tau-t)(n-1)}\frac{L^n(b_{k-l+1},\tau)}{v-b_{k-l+1}}d\tau\\ &+ \int_t^0\int_\tau^0 \lambda^2 e^{-\lambda(s-t)(n-1)}(n-1)\frac{L^n(b_{k-l+2},s)}{v-b_{k-l+1}}dsd\tau\\ &\leq -\lambda t e^{\lambda t(n-1)} + \int_t^0 \lambda(n-2)e^{-\lambda(\tau-t)(n-1)}\frac{L^n(b_{k-l},\tau)}{v-b_{k-l}}d\tau\\ &+ \int_t^0\int_\tau^0 \lambda^2 e^{-\lambda(s-t)(n-1)}(n-1)\frac{L^n(b_{k-l+1},s)}{v-b_{k-l}}dsd\tau\\ &= \frac{L^n(b_{k-l-1},t)}{v-b_{k-l}}.\end{aligned}$$

This proves the lemma. □

Let us now define the cutoff t^l in the following way.

$$e^{\lambda(n-1)t^l}(v-b_{l+1}) = L^n(b_l,t^l).$$

Note that this does not have to be the time at which reversion to truthful bidding by all players is no longer sufficient to deter a losing bidder from bidding, since the continuation value used in the definition is not necessarily equal to the actual continuation value. However, Lemma 4 together with Lemma 3 implies that t^l is decreasing in l for $l \in \{0,...,k-2\}$. This then implies that the continuation value of a losing bidder when price is b_l are correct for the relevant range, that is for times later than $\max_{j\in\{0,...,l\}} t^j$.

This implies that the above cutoff sequence is indeed the one belonging to a maximally delayed equilibrium given bidding sequence S, hence in the latter t^l is decreasing in l for $l \in \{1, ..., k-2\}$.

Given these cutoffs, we can then define the actual continuation values of the winning and losing bidders in the same manner as in the proof of theorem 2.

$$W(b_l, t) = \begin{cases} W^n(b_l, t) & \text{if } t \geq t^l \\ W^n(b_l, t^l) & \text{if } t < t^l. \end{cases}$$

$$L(b_l, t) = \begin{cases} L^n(b_l, t) & \text{if } t \geq t^l \\ L^n(b_l, t^l & \text{if } t < t^l \end{cases}$$

for all $l = 0, 1, \ldots, v-2$. Continuation values at prices b_k and b_{k-1} are defined as usual, $W(b_k, t), L(b_k, t), L(b_{k-1}, t) = 0$ and $W(b_{k-1}, t) = W^n(b_{k-1}, t)$ for all t. Clearly these definitions above rely on the fact that $t^0 \geq t^1 \geq \cdots \geq t^{k-2}$. Furthermore we can define

$$\tilde{L}(\emptyset, t) = \int_t^0 \lambda e^{-\lambda n(\tau - t)} \left(W(0, \tau) + (n-1) L(0, \tau) \right) d\tau.$$

and finally $t^{\emptyset}$:

$$\tilde{L}(\emptyset, t^{\emptyset}) = e^{\lambda(n-1)t^{\emptyset}} v.$$

Then define

$$L(\emptyset, t) = \begin{cases} \tilde{L}(\emptyset, t) & \text{if } t \geq t^{\emptyset} \\ \tilde{L}(\emptyset, t^{\emptyset}) & \text{if } t < t^{\emptyset}. \end{cases}$$

□

The next result establishes that for arbitrary bidding sequences, when the number of bidders is large enough, maximally delayed equilibria again have the property that once a bid is placed, cutoffs are monotonically decreasing in price. This means that when the number of bidders is large, any bidding sequence admits a structure of equilibrium that is qualitatively similar to that of a regular bidding sequence. Therefore as we have shown for regular bidding sequences, the equilibrium path of play in maximally delayed equilibria for any arbitrary sequence with a large number of bidders is such that all of the delay occurs at the beginning of the auction at the prices of $\emptyset$ and 0 (or only at $\emptyset$, if $t^0 \leq t^{\emptyset}$).

Theorem 5. *Let $S = \{b_1, ..., b_k\}$ be a bidding sequence. Then for sufficiently large n, the cutoff sequence belonging to a maximally delayed equilibrium satisfies*

$$t^{\emptyset} > t^0 > t^1 > \cdots > t^{k-2} = -1/\lambda.$$

The proof of this theorem exploits the properties of the functions $L^n(b_l, t)$ and $W^n(b_l, t)$, defined as the continuation of a losing and winning bidder conditional on all players playing according to a gradual bidding strategy profile with no delays.

Let us define the following cutoffs t_n^{k-l}:

$$L^n(b_{k-l}, t_n^{k-l}) = (v - b_{k-l+1})e^{\lambda(n-1)t_n^{k-l}}.$$

Proof: Consider the case in which $l = 2$.

$$L(b_{k-2}, t) = e^{\lambda(n-1)t}(-\lambda t)(v - b_{k-1})$$

This then implies that $t_n^{k-2} = -1/\lambda$ for all n. Consider now $l = 3$.

$$e^{-\lambda(n-1)t}L(b_{k-3}, t) = (-\lambda t)(v - b_{k-2}) + \frac{(-\lambda t)^2}{2}(n-2)(v - b_{k-1})$$

Notice then that $e^{-\lambda(n-1)t}L^n(b_{k-3}, t_n^{k-2}) > v - b_{k-2}$ for all $n > 2$. This means that $t_n^{k-3} > t_n^{k-2}$ for all $n > 2$. Furthermore it is easy to conclude that $t_n^{k-3} \to 0$ from the definition of t_n^{k-3}:

$$(-\lambda t_n^{k-3})(v - b_{k-2}) + \frac{(-\lambda t_n^{k-3})^2}{2}(n-2)(v - b_{k-1}) = v - b_{k-2}.$$

Given the fact that $t_n^{k-3} \to 0$, we must have

$$n(-\lambda t_n^{k-3})^2 \to 2\frac{v - b_{k-2}}{v - b_{k-1}}.$$

Now we induct based on the following inductive hypothesis:

1. $\liminf_{n\to\infty} n^{j-2}(-\lambda t_n^{k-j})^{j-1} > 0$,
2. $t_n^{k-j} \to 0$,
3. and $t_n^{k-2} < t_n^{k-3} < \cdots < t_n^{k-l}$

for all $2 < j \leq l$ and all $n \geq n^*$. As we have shown, the above holds for $l = 3$. Suppose the above holds for some l. Consider the cutoff t_n^{k-l-1}. First note that since $\liminf_{n\to\infty} n^{l-2}(-\lambda t_n^{k-l})^{l-1} > 0$ and $t_n^{k-l} \to 0$,

$$\lim\inf_{n\to\infty} n(-\lambda t_n^{k-l}) = +\infty.$$

Therefore

$$\lim\inf_{n\to\infty} n^{l-1}(-\lambda t_n^{k-l})^l = +\infty.$$

Observe the following expression:

$$\begin{aligned} e^{-\lambda(n-1)t_n^{k-l}}L(b_{k-l-1}, t_n^{k-l}) = &\sum_{j=0}^{l} \frac{(-\lambda t_n^{k-l})^j}{j!}\left(\frac{(n-1)^j}{n} + \frac{(-1)^{j+1}}{n}\right)(v - b_{k-l-1+j}) \\ > &\frac{(-\lambda t_n^{k-l})^l}{l!}\left(\frac{(n-1)^l}{n} + \frac{(-1)^{l+1}}{n}\right)(v - b_{k-1}) \\ \to &+\infty. \end{aligned}$$

This then implies that for n sufficiently large, $t_n^{k-l-1} > t_n^{k-l}$ since $e^{-\lambda(n-1)t}L(b_{k-l-1}, t)$ is a decreasing function due to Lemma 3. Then because of the second hypothesis, this implies that $t_n^{k-l-1} \to 0$. Now consider the following limit:

$$\liminf_{n\to\infty} n^{l-1}(-\lambda t_n^{k-l-1})^l.$$

Suppose that the above is zero. Then it must be that

$$\liminf_{n\to\infty} n^{j-1}(-\lambda t_n^{k-l-1})^j = 0$$

for all $j \leq l$ which implies that

$$\begin{aligned}
(v - b_{k-l}) =& \liminf_{n\to\infty} e^{-\lambda(n-1)t_n^{k-l-1}} L(b_{k-l-1}, t_n^{k-l-1}) \\
=& \liminf_{n\to\infty} \sum_{j=0}^{l} \frac{(-\lambda t_n^{k-l-1})^j}{j!} \left(\frac{(n-1)^j}{n} + \frac{(-1)^{j+1}}{n} \right) (v - b_{k-l-1+j}) \\
=& 0,
\end{aligned}$$

a contradiction. Therefore we must have $\liminf_{n\to\infty} n^{l-1}(-\lambda t_n^{k-l-1})^l > 0$. Lastly, we examine the cutoff when $p = \emptyset$, when no players have bid. This must be treated separately due to the different form that $L(\emptyset, t)$ takes. Define $t_n^{\emptyset}$ as the cutoff when

$$L(\emptyset, t_n^{\emptyset}) = ve^{\lambda(n-1)t_n^{\emptyset}}.$$

Note that

$$\begin{aligned}
e^{-\lambda(n-1)t_n^0} L(\emptyset, t_n^0) &= \sum_{j=0}^{k-1} (-1)^{j+1} \left(\sum_{l=j+1}^{\infty} \frac{(\lambda t_n^0)^l}{l!} \right) (n-1)^j (v - b_j) \\
&\geq (-1)^k \left(\sum_{l=k}^{\infty} \frac{(\lambda t_n^0)^l}{l!} \right) (n-1)^{k-1} (v - b_{k-1}).
\end{aligned}$$

Since $\liminf_{n\to\infty} n^{k-2}(-t_n^0)^{k-1} > 0$ and $t_n^0 \to 0$, we must have

$$\liminf_{n\to\infty} n^{k-1}(-t_n^0)^k = +\infty.$$

Then we know that

$$\begin{aligned}
&(-1)^k \left(\sum_{l=k}^{\infty} \frac{(\lambda t_n^0)^l}{l!} \right) (n-1)^{k-1} (v - b_{k-1}) \\
\geq& (n-1)^{k-1} (-\lambda t_n^0)^k \left(\frac{1}{k!} + \frac{\lambda t_n^0}{(k+1)!} \right) (v - b_{k-1})
\end{aligned}$$

The limit infimum of the last expression then goes to $+\infty$ as $n \to \infty$. Thus

$$\liminf_{n\to\infty} e^{-\lambda(n-1)t_n^0} L(\emptyset, t_n^0) = +\infty.$$

Thus we must have $t_n^{\emptyset} > t_n^0$ for n sufficiently large. In conclusion, we have shown that for n sufficiently large,

$$-1/\lambda = t_n^{k-2} < t_n^{k-3} < \cdots < t_n^0 < t_n^{\emptyset}.$$

□

A.5 Markovian Equilibria with Delays for Large n: Monotonicity of Cutoffs

With the closed form expressions for the value functions given by equations (1), (2), and (3), we can also study how the structure of equilibria characterized in Subsection 4.3 changes as we vary n. As in section A.4, we are particularly interested in the structure of equilibria when the number of bidders is large relative to k, the number of bids in a bid sequence. Let us study cutoffs t_n^{k-l} where

$$W^n(b_{k-l+1}, t_n^{k-l}) = L^n(b_{k-l}, t_n^{k-l})$$

where the value functions above represent the continuation values at time t in an auction with n bidders bidding incrementally over $S = \{b_1, \ldots, b_k\}$ with no delays. To analyze these cutoffs, we use a similar method of proof used in the Section A.4 to study the following expressions for large values of n.

$$W^n(b_{k-l}, t) = e^{\lambda(n-1)t} \sum_{j=0}^{l-1} \frac{(-\lambda t)^j}{j!} \left(\frac{(n-1)^j}{n} + \frac{(-1)^j(n-1)}{n} \right)(v - b_{k-l+j})$$
$$L^n(b_{k-l}, t) = e^{\lambda(n-1)t} \sum_{j=0}^{l-1} \frac{(-\lambda t)^j}{j!} \left(\frac{(n-1)^j}{n} + \frac{(-1)^{j+1}}{n} \right)(v - b_{k-l+j}).$$

Lemma 5. *Let $S = \{b_1, \ldots, b_k\}$ be a bid sequence. Then there exists some n^* such that*

$$t_n^{\emptyset} > t_n^0 > \cdots > t_n^{k-2} = -1/\lambda$$

for all $n > n^*$.

Proof. First we claim that cutoffs above $\emptyset$ are decreasing in price. We proceed by induction. Consider the case in which $l = 2$.

$$W^n(b_{k-1}, t_n^{k-2}) - L^n(b_{k-2}, t_n^{k-2}) = (v - b_{k-1})(1 + \lambda t_n^{k-2})e^{\lambda(n-1)t_n^{k-2}}$$

So $t_n^{k-2} = -1/\lambda$ for all n. But then consider

$$\begin{aligned}
&(W^n(b_{k-2}, t_n^{k-2}) - L^n(b_{k-3}, t_n^{k-2}))e^{-\lambda(n-1)t_n^1} \\
=&(v - b_{k-2})(1 + \lambda t_n^{k-2}) - \frac{(-\lambda t_n^{k-2})^2}{2}(n-2)(v - b_{k-1}) \\
<&0
\end{aligned}$$

when $n > 2$. Thus for sufficiently large n, $t_n^{k-2} < t_n^{k-3}$ due to Corollary 1. Furthermore we can conclude that $t_n^{k-3} \to 0$ and that $\lim_{n\to\infty} n(-\lambda t_n^{k-3})^2 = (v - b_{k-2})/(v - b_{k-1}) > 0$. Now we induct. Suppose that there exists some n^* such that

1. $\liminf_{n\to\infty} n^{j-2}(-\lambda t_n^{k-j})^{j-1} > 0$,
2. $\limsup_{n\to\infty} n^{j-2}(-\lambda t_n^{k-j})^{j-1} < \infty$,
3. $t_n^{k-j} \to 0$,
4. and $t_n^{k-2} < t_n^{k-3} < \cdots < t_n^{k-l}$

for all $2 < j \leq l$ and all $n \geq n^*$. Obviously, as we have already shown, all of the assumptions above hold for $l = 3$. Suppose the above hypotheses hold for $l \geq 3$. Consider the cutoff t_n^{k-l-1}. Then

$$\liminf_{n\to\infty} \frac{n^{l-2}(-\lambda t_n^{k-l})^{l-1}}{t_n^l} = \liminf_{n\to\infty} (-\lambda n t_n^{k-l})^{l-2} = +\infty.$$

Thus

$$\lim_{n\to\infty} -\lambda n t_n^{k-l} = +\infty$$

which implies

$$\lim_{n\to\infty} n^{l-1}(-\lambda t_n^{k-l})^l = +\infty.$$

Then consider the expression

$$e^{-\lambda(n-1)t_n^{k-l}} \left(W^n(b_{k-l}, t_n^{k-l}) - L^n(b_{k-(l+1)}, t_n^{k-l})\right)$$

We first show that the above is negative for sufficiently large n. The above expression is

$$\sum_{j=0}^{l-1} \frac{(-\lambda t_n^{k-l})^j}{j!} \left(\frac{(n-1)^j}{n} + \frac{(-1)^j(n-1)}{n}\right)(v - b_{k-l+j})$$
$$-\sum_{j=0}^{l} \frac{(-\lambda t_n^{k-l})^j}{j!} \left(\frac{(n-1)^j}{n} + \frac{(-1)^{j+1}}{n}\right)(v - b_{k-l+j-1})$$

The above is less than or equal to

$$v - b_{k-l} + \sum_{j=2}^{l-1} \frac{(-\lambda t_n^{k-(j+1)})^j}{j!} \left(\frac{(n-1)^j}{n} + \frac{(-1)^j(n-1)}{n}\right)(v - b_{k-l+j})$$
$$-\frac{(-\lambda t_n^{k-l})^l}{l!} \left(\frac{(n-1)^l}{n} + \frac{(-1)^{l+1}}{n}\right)(v - b_{k-1})$$

for sufficiently large n. But by the inductive hypothesis the above goes to $-\infty$ as $n \to \infty$. This then implies that for sufficiently large n,

$$W^n(b_{k-l}, t_n^{k-l}) - L^n(b_{k-l-1}, t_n^{k-l}) < 0.$$

Therefore $t_n^{k-l-1} > t_n^{k-l}$ for sufficiently large n due to Corollary 1. Now we need to prove the other statements used in the inductive hypothesis. Clearly since $t_n^{k-l} \to 0$, we must have $t_n^{k-l-1} \to 0$. Consider

$$\liminf_{n\to\infty} n^{l-1}(-\lambda t_n^{k-l-1})^l.$$

Suppose that this expression is 0. Then

$$0 = \liminf_{n\to\infty} n^{\frac{(l-1)^2}{l}}(-\lambda t_n^{k-l-1})^{l-1} \geq \liminf_{n\to\infty} n^{l-2}(-\lambda t_n^{k-l-1})^{l-1}$$

Iterating the argument we obtain

$$\liminf_{n\to\infty} n^{j-1}(-\lambda t_n^{k-l-1})^j = 0$$

for all $1 \leq j \leq l$. By definition

$$\begin{aligned}&\sum_{j=0}^{l-1} \frac{(-\lambda t_n^{k-l-1})^j}{j!}\left(\frac{(n-1)^j}{n} + \frac{(-1)^j(n-1)}{n}\right)(v - b_{k-l+j})\\ &-\sum_{j=0}^{l} \frac{(-\lambda t_n^{k-l-1})^j}{j!}\left(\frac{(n-1)^j}{n} + \frac{(-1)^{j+1}}{n}\right)(v - b_{k-l+j-1}) = 0\end{aligned}$$

for all n. But taking the limit infimum of both sides of the equation above implies that $(v - b_{k-l}) = 0$, which is a contradiction. So we must have

$$\liminf_{n\to\infty} n^{l-1}(-\lambda t_n^{k-l-1})^l > 0.$$

Showing that the limit supremum is finite can be derived in a similar manner. This concludes the induction. Note that the above argument implies some other useful facts:

1. $0 < \liminf_{n\to\infty} n^{j-2}(-\lambda t_n^{k-j})^{j-1} \leq \limsup_{n\to\infty} n^{j-2}(-\lambda t_n^{k-j})^{j-1} < +\infty$,
2. $t_n^{k-j} \to 0$,
3. and there exists some n^* such that $-1/\lambda = t_n^{k-2} < t_n^{k-3} < \cdots < t_n^0$ for all $n \geq n^*$,

for all $j = 3, \ldots, k$. This will be used below, for analyzing the case of $p = \emptyset$. Next we show that monotonicity extends to the cutoff belonging to $\emptyset$. Because the functional form of $L(\emptyset, t)$ is a bit different from the other value functions for losing bidders, it must be treated independently. However a similar argument can be used to characterize the qualitative nature of the cutoff $t_n^{\emptyset}$ where

$$W^n(0, t_n^{\emptyset}) = L^n(\emptyset, t_n^{\emptyset}).$$

Consider the expression:

$$e^{-\lambda(n-1)t_n^0} L^n(\emptyset, t_n^0) \geq (-1)^k \left(\sum_{l=k}^{\infty} \frac{(\lambda t_n^0)^l}{l!} \right) (n-1)^{k-1}(v - b_{k-1})$$
$$= (n-1)^{k-1}(v - b_{k-1}) \frac{(-\lambda t_n^0)^k}{k!} \sum_{l=k}^{\infty} \frac{(\lambda t_n^0)^{l-k} k!}{l!}$$

Note that

$$\sum_{l=k}^{\infty} \frac{(\lambda t_n^0)^{l-k} k!}{l!} > 1 + \frac{\lambda t_n^0}{k+1} \to 1$$

as $n \to \infty$. Furthermore we know that

$$0 < \liminf_{n\to\infty} n^{k-2}(-t_n^0)^{k-1}$$

and that $t_n^0 \to 0$. Therefore

$$\liminf_{n\to\infty} n^{k-1}(-t_n^0)^k = +\infty.$$

Together all of the above observations above imply that

$$\liminf_{n\to\infty} e^{-\lambda(n-1)t_n^0} L^n(\emptyset, t_n^0) = +\infty.$$

But we can also show that $\limsup_{n\to\infty} e^{-\lambda(n-1)t_n^0} W^n(0, t_n^0) < +\infty$. Using the closed form expressions for the value functions again, we have

$$e^{-\lambda(n-1)t_n^0} W^n(0, t_n^0) = \sum_{j=0}^{k-1} \frac{(-\lambda t_n^0)^j}{j!} \left(\frac{(n-1)^j + (-1)^j(n-1)}{n} \right) (v - b_j)$$
$$\leq (v - b_0) + \sum_{j=2}^{k-1} \frac{(-\lambda t_n^{k-j-1})^j}{j!} \left(\frac{(n-1)^j + (-1)^j(n-1)}{n} \right) (v - b_j)$$

Then taking the limit supremum of both sides and using the fact that

$$\limsup_{n\to\infty} n^{j-1}(-t_n^{k-j-1})^j < +\infty$$

for all $j = 2, \ldots, k-1$ immediately implies that

$$\limsup_{n\to\infty} e^{-\lambda(n-1)t_n^0} W^n(0, t_n^0) < +\infty.$$

This implies that $t_n^\emptyset > t_n^0$ for n sufficiently large. Thus we have shown that for n sufficiently large, we must have

$$t_n^{k-2} < t_n^{k-3} < \cdots < t_n^0 < t_n^\emptyset.$$

$\square$

B Comparative Statics

B.1 Changes in v

Let us first derive a useful expression for the seller's expected revenue:

$$\begin{aligned}
&\int_{t_v^{\emptyset}}^{0} \lambda n e^{-\lambda n(\tau - t_v^{\emptyset})} \left(v - (n-1)L(0,\tau) - W(0,\tau)\right) d\tau \\
&= \left(1 - e^{\lambda n t_v^{\emptyset}}\right) v - nL(\emptyset, t_v^{\emptyset}) \\
&= v\left(1 - e^{\lambda n t_v^{\emptyset}} - n e^{\lambda(n-1)t_v^{\emptyset}}\right).
\end{aligned}$$

Before proceeding further, we first prove a lemma that provides some useful properties of relevant value functions. Define the following functions $\hat{W}^v(0,\tau)$ and $\hat{L}^v(0,\tau)$ as the continuation values of a winning and losing bidder at a current price of 0 and current winning bid of 1 with evaluation v in which players play a completely gradual bidding strategy with no delays at all times $\tau' > \tau$.

Lemma 6. *$\hat{W}^v(0,\tau) + (n-1)\hat{L}^v(0,\tau)$ is increasing in τ.*

Proof. Note that

$$\hat{W}^v(0,\tau) + (n-1)\hat{L}^v(0,\tau) = e^{\lambda(n-1)\tau} \sum_{k=0}^{v} \frac{(-\lambda(n-1)\tau)^k}{k!}(v-k).$$

But observe that the function

$$\tau \mapsto e^{\tau} \sum_{k=0}^{v} \frac{(-\tau)^k}{k!}(v-k)$$

is increasing in τ when $\tau < 0$ since the derivative with respect to τ of the above is given by

$$\begin{aligned}
&e^{\tau} \sum_{k=0}^{v} \frac{(-\tau)^k}{k!}(v-k) - e^{\tau} \sum_{k=1}^{v} \frac{(-\tau)^{k-1}}{(k-1)!}(v-k) \\
&= e^{\tau} \sum_{k=0}^{v} \frac{(-\tau)^k}{k!}(v-k) - e^{\tau} \sum_{k=0}^{v-1} \frac{(-\tau)^k}{k!}(v-k-1) \\
&= e^{\tau} \sum_{k=0}^{v-1} \frac{(-\tau)^k}{k!} > 0.
\end{aligned}$$

Thus $\hat{W}^v(0,\tau) + (n-1)\hat{L}^v(0,\tau)$ is also increasing in τ. $\square$

Lemma 7. *The following hold for the maximally delayed equilibrium corresponding to the completely gradual bidding sequence: (i) t_v^0 increases as v increases, and it converges to a limit strictly below* 0 *as $v \to \infty$; (ii) similarly, $t_v^\emptyset$ is increasing in v and converges to a limit strictly below* 0 *as $v \to \infty$.*

Proof. Recall that t_v^0 is defined by:

$$e^{\lambda(n-1)t_v^0}(v-1) = L(0, t_v^0).$$

Therefore

$$1 = \sum_{j=0}^{v-1} \frac{(-\lambda t_v^0)^j}{j!} \left(\frac{(n-1)^j + (-1)^{j+1}}{n} \frac{(v-j)}{v-1} \right).$$

Note that the right hand side is a decreasing function in t_v^0 and increasing in v. Hence, t_v^0 must be increasing in v. Next, note that

$$\begin{aligned} L(0, t_v^0) &= \int_{t_v^0}^{0} \lambda(n-1)e^{-\lambda(n-1)(\tau - t_v^0)} \left(\frac{1}{n-1} W(1,\tau) + \frac{n-2}{n-1} L(1,\tau) \right) d\tau \\ &< \int_{t^0}^{0} \lambda(n-1)e^{-\lambda(n-1)(\tau - t_v^0)} \frac{v-1}{n-1} d\tau \\ &= \left(1 - e^{\lambda(n-1)t_v^0}\right) \frac{v-1}{n-1}. \end{aligned}$$

Thus

$$\frac{L(0, t_v^0)}{v-1} < \left(1 - e^{\lambda(n-1)t_v^0}\right) \frac{1}{n-1}.$$

If $t_v^0 \to 0$ as $v \to \infty$, then the above inequality would imply that

$$\frac{L(0, t_v^0)}{v-1} \to 0.$$

However by definition, we had

$$\frac{L(0, t_v^0)}{v-1} = e^{\lambda(n-1)t_v^0} \to 1$$

which is a contradiction. We now use the monotonicity of the cutoff $t_v^\emptyset$ in v to show that $t_v^\emptyset < t_{v+1}^\emptyset$. Define the function $\tilde{L}^{v+1}(\emptyset, t_v^\emptyset)$ as the continuation payoff to a player when all players play according to the equilibrium strategy when the evaluation is v when in reality the value is $v+1$. Now suppose that $t_{v+1}^\emptyset \le t_v^\emptyset$. Then note that

$$\tilde{L}^{v+1}(\emptyset, t_v^\emptyset) = \int_{t_v^\emptyset}^{0} \lambda n e^{-\lambda n(\tau - t_v^\emptyset)} \left(\hat{W}^{v+1}(0, \max\{\tau, t_v^0\}) + (n-1)\hat{L}^{v+1}(0, \max\{\tau, t_v^0\})\right)$$

whereas

$$L^{v+1}(\emptyset, t_v^\emptyset) = \int\limits_{t_v^\emptyset}^{0} \lambda n e^{-\lambda n(\tau - t_v^\emptyset)} \left(\hat{W}^{v+1}(0, \max\{\tau, t_{v+1}^0\}) + (n-1)\hat{L}^{v+1}(0, \max\{\tau, t_{v+1}^0\})\right).$$

Because $t_v^0 < t_{v+1}^0$ and using Lemma 6, observe that $\tilde{L}^{v+1}(\emptyset, t_v^\emptyset) < L^{v+1}(\emptyset, t_v^\emptyset)$. We now consider the values $\tilde{L}^{v+1}(\emptyset, t_v^\emptyset)/(v+1)$ and $L^v(\emptyset, t_v^\emptyset)/v$. Under the strategy corresponding to the value function $\tilde{L}^{v+1}(\emptyset, t_v^\emptyset)$ players play exactly the same way as in the strategy corresponding to $L^v(\emptyset, t_v^\emptyset)$. Then conditional on any sequence of arrivals, if the outcome is such that he does not win under the $\tilde{L}$ strategy, he will also not win in the equilibrium corresponding to the value function L. If instead conditional on a sequence of arrivals, the payoff of the bidder is strictly positive, then the outcome payoff divided by $v+1$ is $(x+1)/(v+1)$ whereas the outcome payoff in the L equilibrium is x/v. But note that $(x+1)/(v+1) > x/v$. Thus we must have $\tilde{L}^{v+1}(\emptyset, t_v^\emptyset)/(v+1) > L^v(\emptyset, t_v^\emptyset)/v$ since the two strategies induce the same probability measure over arrival sequences. This means that

$$\frac{L^{v+1}(\emptyset, t_v^\emptyset)}{v+1} \geq \frac{\tilde{L}^{v+1}(\emptyset, t_v^\emptyset)}{v+1} > \frac{L^v(\emptyset, t_v^\emptyset)}{v} = e^{\lambda(n-1)t_v^\emptyset}.$$

Thus $t_{v+1}^\emptyset > t_v^\emptyset$ and we have established the monotonicity claim. Finally we show that $\lim_{v\to\infty} t_v^\emptyset < 0$. The proof is very similar to the proof for the case of t_v^0. Suppose that $t_v^\emptyset \to 0$. Then

$$L(\emptyset, t_v^\emptyset) \leq \left(1 - e^{\lambda n t_v^\emptyset}\right) v$$

and so

$$\frac{L(\emptyset, t_v^\emptyset)}{v} \leq \left(1 - e^{\lambda n t_v^\emptyset}\right) \to 0.$$

However $e^{\lambda(n-1)t_v^\emptyset} \to 1$ but this is a contradiction since by definition $L(\emptyset, t_v^\emptyset) = ve^{\lambda(n-1)t_v^\emptyset}$. This concludes the proof. □

Proof of Claim 3: Below we compare the seller's expected revenue at $v+1$ and v. Let us define the continuation value function $\tilde{L}^v(\emptyset, t_{v+1}^\emptyset)$ as the continuation value function of a bidder conditional on there having been no bids and all bidders following the strategy corresponding to the equilibrium with value $v+1$ after time $t_{v+1}^\emptyset$ when in reality all bidders have valuation v. First we consider the case in which $t_v^\emptyset < t_{v+1}^\emptyset$. Using the expression for the expected revenue of a seller, we can write the difference between the expected revenue at consumer evaluation of $v+1$ and the expected revenue at v:

$$\begin{aligned}
\Delta\Pi_v &\equiv (1 - e^{\lambda n t_{v+1}^\emptyset})(v+1) - nL^{v+1}(\emptyset, t_{v+1}^\emptyset) - (1 - e^{\lambda n t_v^\emptyset})v + nL^v(\emptyset, t_v^\emptyset) \\
&= (1 - e^{\lambda n t_{v+1}^\emptyset}) + v(e^{\lambda n t_v^\emptyset} - e^{\lambda n t_{v+1}^\emptyset}) - n(L^{v+1}(\emptyset, t_{v+1}^\emptyset) - L^v(\emptyset, t_v^\emptyset)) \\
&= (1 - e^{\lambda n t_{v+1}^\emptyset}) + v(e^{\lambda n t_v^\emptyset} - e^{\lambda n t_{v+1}^\emptyset}) - n(L^{v+1}(\emptyset, t_{v+1}^\emptyset) - \tilde{L}^v(\emptyset, t_{v+1}^\emptyset)) + n(L^v(\emptyset, t_v^\emptyset) - \tilde{L}^v(\emptyset, t_{v+1}^\emptyset)) \\
&= -ve^{\lambda n t_{v+1}^\emptyset}(1 - e^{-\lambda n(t_{v+1}^\emptyset - t_v^\emptyset)}) + n(L^v(\emptyset, t_v^\emptyset) - \tilde{L}^v(\emptyset, t_{v+1}^\emptyset)).
\end{aligned}$$

Then note that

$$nL^v(\emptyset, t^{\emptyset}_{v+1}) = \int_{t^{\emptyset}_{v+1}}^{0} \lambda n e^{-\lambda n(\tau - t^{\emptyset}_v)} \left(\hat{W}^v(0, \max\{\tau, t^0_v\}) + (n-1)\hat{L}^v(0, \max\{\tau, t^0_v\}) \right) d\tau.$$

On the other hand,

$$n\tilde{L}^v(\emptyset, t^{\emptyset}_{v+1}) = \int_{t^{\emptyset}_{v+1}}^{0} \lambda n e^{-\lambda n(\tau - t^{\emptyset}_v)} \left(\hat{W}^v(0, \max\{\tau, t^0_{v+1}\}) + (n-1)\hat{L}^v(0, \max\{\tau, t^0_{v+1}\}) \right) d\tau.$$

Then from Lemma 6 and because $t^0_v < t^0_{v+1}$, we have $nL^v(\emptyset, t^{\emptyset}_{v+1}) \leq n\tilde{L}^v(\emptyset, t^{\emptyset}_{v+1})$. Therefore, we can conclude that

$$\Delta\Pi_v \leq -ve^{\lambda n t^{\emptyset}_{v+1}}(1 - e^{-\lambda n(t^{\emptyset}_{v+1} - t^{\emptyset}_v)}) + n(L^v(\emptyset, t^{\emptyset}_v) - L^v(\emptyset, t^{\emptyset}_{v+1})).$$

Then to show that $\Delta\Pi_v \leq 0$, it is sufficient to show that

$$n(L^v(\emptyset, t^{\emptyset}_v) - L^v(\emptyset, t^{\emptyset}_{v+1})) \leq ve^{\lambda n t^{\emptyset}_{v+1}}(1 - e^{-\lambda n(t^{\emptyset}_{v+1} - t^{\emptyset}_v)}). \tag{4}$$

Because $t^{\emptyset}_v < t^{\emptyset}_{v+1}$, we can rewrite $nL^v(\emptyset, t^{\emptyset}_v)$ in the following way:

$$\begin{aligned}
nL^v(\emptyset, t^{\emptyset}_v) &= e^{-\lambda n(t^{\emptyset}_{v+1} - t^{\emptyset}_v)} nL^v(\emptyset, t^{\emptyset}_{v+1}) \\
&\quad + \int_{t^{\emptyset}_v}^{t^{\emptyset}_{v+1}} \lambda n e^{-\lambda n(\tau - t^{\emptyset}_v)} \left(\hat{W}^v(0, \max\{\tau, t^0_v\}) + (n-1)\hat{L}^v(0, \max\{\tau, t^0_v\}) \right) d\tau \\
&\leq e^{-\lambda n(t^{\emptyset}_{v+1} - t^{\emptyset}_v)} nL^v(\emptyset, t^{\emptyset}_{v+1}) + \left(1 - e^{-\lambda n(t^{\emptyset}_{v+1} - t^{\emptyset}_v)}\right) \left(\hat{W}^v(0, t^*) + (n-1)\hat{L}^v(0, t^*) \right)
\end{aligned}$$

where $t^* = \max\{t^{\emptyset}_{v+1}, t^0_v\}$ and the last inequality again uses Lemma 6. Therefore inequality (4) holds if

$$\hat{W}^v(0, t^*) + (n-1)\hat{L}^v(0, t^*) \leq ve^{\lambda n t^{\emptyset}_{v+1}} + nL^v(\emptyset, t^{\emptyset}_{v+1}). \tag{5}$$

First if $t^* = t^{\emptyset}_{v+1}$, then the above holds since

$$\begin{aligned}
ve^{\lambda n t^{\emptyset}_{v+1}} + nL^v(\emptyset, t^{\emptyset}_{v+1}) &\geq ve^{\lambda n t^{\emptyset}_{v+1}} + \left(1 - e^{\lambda n t^{\emptyset}_{v+1}}\right) \left(\hat{W}^v(0, t^{\emptyset}_{v+1}) + (n-1)\hat{L}^v(0, t^{\emptyset}_{v+1}) \right) \\
&\geq \hat{W}^v(0, t^{\emptyset}_{v+1}) + (n-1)\hat{L}^v(0, t^{\emptyset}_{v+1}).
\end{aligned}$$

On the other hand if $t^* > t^{\emptyset}_{v+1}$, then the above inequality also holds because

$$\begin{aligned}
ve^{\lambda n t^{\emptyset}_{v+1}} + nL^v(\emptyset, t^{\emptyset}_{v+1}) &= ve^{\lambda n t^{\emptyset}_{v+1}} + \left(1 - e^{-\lambda n(t^0_v - t^{\emptyset}_{v+1})}\right) \left(\hat{W}^v(0, t^0_v) + (n-1)\hat{L}^v(0, t^0_v) \right) \\
&\quad + e^{-\lambda n(t^0_v - t^{\emptyset}_{v+1})} nL^v(\emptyset, t^0_v).
\end{aligned}$$

Thus inequality (5) holds if

$$\hat{W}^v(0,t_v^0) + (n-1)\hat{L}^v(0,t_v^0) \leq ve^{\lambda n t_v^0} + nL^v(\emptyset, t_v^0).$$

Again from Lemma 6, we can conclude that

$$\begin{aligned} ve^{\lambda n t_v^0} + nL^v(\emptyset, t_v^0) &\geq ve^{\lambda n t_v^0} + \left(1 - e^{\lambda n t_v^0}\right)\left(\hat{W}^v(0,t_v^0) + (n-1)\hat{L}^v(0,t_v^0)\right) \\ &> \hat{W}^v(0,t_v^0) + (n-1)\hat{L}^v(0,t_v^0). \end{aligned}$$

Thus we have shown that for a fixed n, $\Delta\Pi_v \leq 0$ for all v. This concludes the proof of the monotonicity result. Because expected revenue is decreasing in v and is always positive, it must trivially converge to some $\pi^* \geq 0$. The reason that π^* is strictly positive is that by Lemma 7, $\lim_{v\to\infty} \max\{t_v^{\emptyset}, t_v^0\} < 0$. Therefore the probability of the price reaching 1 cannot converge to zero and so limiting expected revenue π^* must be strictly positive. This concludes the proof. □

C Extensions

C.1 Asymmetric Values

Proof of Proposition 1: We construct an equilibrium characterized by two cutoff points, $t_{\emptyset}^H$ and $t_0^H > t_{\emptyset}^H$. The low type's equilibrium strategy is to bid 1, whenever $p = \emptyset$ and to bid v_L whenever $p \geq 0$. The high type's equilibrium strategy is to bid v_H iff $p = \emptyset$ and $t \geq t_{\emptyset}^H$ or $p = 0$ and $t \geq t_0^H$. We begin by solving for the high type's cutoff times. Given the strategies described above, the high type's continuation values at $p = 0, 1$ are given by,

$$\begin{aligned} W_H(p,t) &= v_H - v_L + (v_L - p)e^{\lambda t} \\ L_H(0,t) &= \int_t^0 \lambda e^{-\lambda(\tau - t)} W_H(1,\tau)d\tau \\ &= (v_H - v_L)(1 - e^{\lambda t}) - \lambda t(v_L - 1)e^{\lambda t}. \end{aligned}$$

When $p = \emptyset$, at $t_{\emptyset}^H$ the high type is indifferent between making a bid of v_L and delaying bidding. Therefore, $t_{\emptyset}^H$ must satisfy

$$W_H(0, t_{\emptyset}^H) = \int_{t_{\emptyset}^H}^0 \lambda e^{-2\lambda(\tau - t_{\emptyset}^H)}\left(W_H(0,\tau) + L_H(0,\tau)\right)d\tau.$$

Note that such a $t_{\emptyset}^H$ must exist for the following reason. The value functions are given by the following expressions:

$$\begin{aligned} W_H(0, t_{\emptyset}^H) &= (v_H - v_L) + v_L e^{\lambda t_{\emptyset}^H}, \\ L_H(\emptyset, t_{\emptyset}^H) &= (v_H - v_L + e^{\lambda t_{\emptyset}^H})(1 - e^{\lambda t_{\emptyset}^H}) - e^{\lambda t_{\emptyset}^H}\lambda t_{\emptyset}^H (v_L - 1). \end{aligned}$$

It follows that:

$$e^{-\lambda t}\left(W_H(0,t) - L_H(\emptyset,t)\right) = v_H + \lambda t(v_L - 1) - (1 - e^{\lambda t}) \tag{6}$$

which is clearly increasing in t. Therefore a unique $t^H_\emptyset$ exists since $t^H_\emptyset$ is a root of the expression above. Thus this shows that $W_H(0,t) > L_H(\emptyset,t)$ for all $t > t^H_\emptyset$ and $W_H(0,t^H_\emptyset) \leq L_H(\emptyset,t^H_\emptyset)$ for all $t \leq t^H_\emptyset$, establishing incentive compatibility for the high type at all times at a price of $\emptyset$. Similarly, because the low type bidder does not raise his bid while holding the current high bid at $p = 0$,

$$\begin{aligned} W_H(1,t^H_0) &= \int_{t^H_0}^{0} \lambda e^{-\lambda(\tau - t^H_0)} W_H(1,\tau)d\tau \\ &= (v_H - v_L)(1 - e^{\lambda t^H_0}) - \lambda t^H_0(v_L - 1)e^{\lambda t^H_0} \end{aligned} \tag{7}$$

which yields $t^H_0 = -\frac{1}{\lambda}\frac{v_H-1}{v_L-1}$. Note again that this is unique since

$$e^{-\lambda t}(W_H(1,t) - L_H(0,t)) = v_H + \lambda t(v_L - 1) - 1$$

is strictly increasing in t. By comparing expressions (6) and (7), we can see that $t^\emptyset_H < t^0_H$. Moreover we have shown that $W_H(1,t) > L_H(0,t)$ for all $t > t^0_H$ and $W_H(1,t) \leq L_H(0,t)$ for all $t \leq t^0_H$, again establishing incentive compatibility for the high type a t a price of 0. Now consider the incentives of the low type. At a price of $p = 0$ when the low type is the losing bidder, the highest bid is already v_L and so the low type bidder is indifferent between placing a bid of v_L or not bidding at all. Next consider a price of $p = \emptyset$. We show that the low type strictly prefers to bid 1 immediately upon arrival at all times at a price of $p = \emptyset$. When $t \geq t^H_0$, the high type's strategy going forward is independent of the low type's actions and therefore the low type must strictly prefer to place a bid, as it strictly increases the likelihood of winning without affecting the expected payoff from winning. Therefore the continuation value to the low type of being a winning bidder at time t^H_0 and a price of $p = 0$ must be strictly bigger than the continuation value to being a losing bidder at a price of $p = \emptyset$:

$$W_L(0,t^H_0) > L_L(\emptyset,t^H_0) > 0.$$

But now note that the continuation value to being a winning bidder at a price of $p = 0$ is equal to $W_L(0,t^H_0)$ at all times $t < t^H_0$ since the high type only becomes active after time t^H_0. But note that the continuation value to being a losing bidder at $p = \emptyset$ for the low type is a convex combination of $L_L(\emptyset,t^H_0)$, 0, and $W_L(0,t^H_0)$ with strictly positive probability on $L_L(\emptyset,t^H_0)$ and 0. This however means that $W_L(0,t)$ must be strictly greater than $L_L(\emptyset,t)$ at all times. Therefore this establishes that the low type's best response at $\emptyset$ is to play v_L upon arrival. This concludes the proof. $\square$

C.2 Asymmetric Information

Proof of Proposition 2: Consider the following strategies: for $t < t^* < 0$, high types bid v_L whenever $p = \emptyset$, they do not make a bid at $p = 0$ and bid v_H

whenever $p > 0$. For $t \geq t^*$, high types that do not hold the high bid place a bid of v_H at any price and high types holding the highest bid do not place bids. Low types bid v_L if $p = \emptyset$ and $t < t^*$, and bid v_L if they do not hold the winning bid, $t \geq t^*$ and $p < v_L$. It follows immediately that the above strategies are best responses to each other for $t > t^*$. A high type bidder that holds the high bid does not benefit from increasing her bid as she has outbid the low type and cannot outbid a high type. Similarly, a low type with a high bid cannot benefit from raising their bid. We now solve for t^* and show that strategies are optimal for $t < t^*$. Let $v_q(t)$ denote the expected value of a high type (who does not currently hold the highest bid) who decides to make a bid at time t and $p = 0$ given bidders follow the above strategy in the future. Then

$$v_q^n(t) = (v_H - v_L)\left(q + (1-q)e^{\lambda t}\right)^{n-1}.$$

Note that $v_q^n(t)$ is an increasing function of t. Now let $V_q^n(t)$ denote the expected continuation value of a high type at time t and price 0 (who does not currently hold the highest bid) who plans to bid upon the next arrival according to the strategy specified above given that there are $n-1$ other players each with probability q of being a low type.

$$\begin{aligned} V_q^n(t) &= \int_t^0 \lambda(n-1)e^{-\lambda(n-1)(\tau-t)}\left(\frac{1}{n-1}v_q^n(\tau) + \frac{n-2}{n-1}qV_q^{n-1}(\tau)\right)d\tau \\ &= \int_t^0 \lambda e^{-\lambda(n-1)(\tau-t)}\left(v_q^n(\tau) + (n-2)qV_q^{n-1}(\tau)\right)d\tau.^{19} \end{aligned}$$

We can then solve for $V_q^n(t)$ by inducting on n to show that

$$V_q^n(t) = (v_H - v_L)\left(q + (1-q)e^{\lambda t}\right)^{n-2}\left(q - e^{\lambda t}\left(q + (1-q)\lambda t\right)\right).$$

Thus

$$V_q^n(t^*) = v_q^n(t^*) \quad \Leftrightarrow \quad t^* = -\frac{1}{\lambda(1-q)}.$$

It is easy to check that $V_q(t) > v_q(t)$ for all $t < t^*$ and that $V_q(t) > v_q(t)$ for all $t > t^*$. Therefore we have checked the high type's incentives at a price of $p = 0$. For a losing low type bidder at a price of 0, he is completely indifferent between bidding and not bidding. Thus his strategy is incentive compatible. Let us now check incentives at a price $p > 0$. Prices $p \neq 0, v_L, v_H$ do not occur on the equilibrium path and so we can specify beliefs to place probability one on the winning opponent being a high type and having bid v_H at such histories. Therefore all players are completely indifferent between bidding and not bidding at such a history. If $p = v_L, v_H$, then the low type is again completely indifferent

[19] Note that if a high type arrives before the bidder obtains an arrival, then the bidder obtains a payoff of 0 and so those events do not enter into the integral.

between bidding and not bidding. For the high type, if the price is $p = v_L$, he weakly prefers to bid v_H today regardless of his beliefs about the highest bid. If the price is $p = v_H$, then he is again indifferent. Finally, if $p = \emptyset$, then both types are completely indifferent between any positive bid that they place since at a price of 0, players do not bid until time t^* and simply bid their valuation after $t \geq t^*$. Furthermore all players strictly prefer to become a winning bidder at all times $t \leq t^*$ than not placing a bid since delaying bidding only increases the chance of never becoming a winning bidder without any additional benefits. $\square$