Local Risk-Sharing Agreements

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Abstract

This paper considers the effect of local information constraints on informal risk sharing, that is when agents can only condition bilateral transfer arrangements on endowment realizations of each other and common friends. We derive necessary and sufficient conditions for Pareto efficient risk-sharing arrangements subject to these constraints, for general networks, utilities, and endowment distributions. These conditions are generalizations of the classic Borch rule, to an expectational form. We provide a more explicit characterization of the optimal arrangements in the case of CARA utilities and normally distributed endowments. With independent endowments optimal transfer rules are particularly simple: each agent distributes her endowment shock equally among her direct neighbors and herself. In our model more central agents become quasi insurance providers to more peripheral agents and end up with higher consumption variance, providing a testable implication that contrasts predictions from models that explain imperfect risk sharing via enforcement constraints. The model can also explain why informal insurance can work much better in one setting than in another one, even if the social network and shock distributions are similar.

Keywords:

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1 Introduction

Informal insurance via the social network has been shown to play an important role in consumption smoothing, and particularly in weathering the impacts of negative income shocks, under various settings (Rosenzweig (1988), Deaton (1992), Paxson (1993), Udry (1994), Townsend (1994), Grimard (1997), Fafchamps and Lund (2003) and Fafchamps and Gubert (2007a)). The main finding in most of these papers is that such informal insurance achieves fairly good but imperfect consumption smoothing. There are different theoretical explanations as to why perfect risk sharing is not possible. One leading explanation is the presence of enforcement constraints: since risk-sharing agreements are informal, they have to satisfy

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incentive compatibility, implying that there exists an upper bound on the amount of transfer that agents can credibly promise to each other. This type of explanation has been explored in a social network framework by Ambrus, Mobius, and Szeidl (2014, AMS from now on).

In this paper we explore an explanation featuring local information constraints: agents can only observe endowment realizations of their direct neighbors, and insurance agreements between two connected agents can only be conditioned on local information, that is, the common information they possess, consisting of the endowment realizations of the two of them and the endowment realizations of their common neighbors. In contrast, existing models of informal risk sharing on networks (Bramoullé and Kranton (2007), Ambrus, Mobius, and Szeidl (2014), Ambrus, Chandrasekhar, and Elliott (2015)) assume that any bilateral agreement between connected agents can be conditioned on global information, or every individual’s endowment realization in the community.

We find that this explanation generates qualitatively different predictions, which are empirically testable, relative to models of informal insurance with enforcement constraints. Hence, our results can help future empirical work to identify which type of constraints plays the key role in keeping informal insurance agreements from the efficient frontier.

There is a line of theoretical literature investigating the effect of imperfect observability of incomes in informal risk sharing, not on general networks but between just two agents in isolation: see for example Townsend (1982), Thomas and Worrall (1990), and Wang (1995). The questions investigated in this literature are fundamentally different from the ones we focus on, mainly because we are interested in questions that are inherently network related, while the above papers focus on just a pair of agents.

The first part of our analysis characterizes Pareto efficient risk-sharing arrangements under local information constraints for general (concave) utility functions and endowment distributions. In the standard framework of risk-sharing arrangements that can use global information, the necessary and sufficient condition for Pareto optimality can be derived relatively easily, as in each state transfers can be modified in arbitrary ways, independently of how transfers are specified in all other states. This leads to a simple necessary and sufficient condition for optimality, referred to as the Borch rule (Borch (1962), Wilson (1968)).

1See also Karlan, Mobius, Rosenblat, and Szeidl (2009), who investigate enforcement constraints in the case of a single borrowing transaction. There is also an extensive literature on the effects of limited commitment on risk-sharing possibilities for a pair of agents instead of general networks (Coate and Ravallion (1993), Kocherlakota (1996), Ligon (1998). Falchamps (1999), Ligon, Thomas, and Worrall (2002), Dubois, Jullien, and Magnac (2008)).

Bloch, Genicot, and Ray (2008) consider various exogenously specified transfer rules that do not have to depend on all endowment realizations, but a transfer between two agents does depend on transfers made to or received from other agents - that is on nonlocal information. For example, global equal sharing at the component level is feasible in the framework of Bloch, Genicot, and Ray (2008), while it is not for general networks in our model. See also Bourlès, Bramoullé, and Perez-Richet (2016), where agents are motivated to send transfers to their neighbors for explicit altruistic reasons, but like in Bloch, Genicot, and Ray (2008), bilateral transfers depend on transfers among other agents.

Empirical papers trying to distinguish among different reasons of imperfectness of informal insurance contracts include Kinnan (2011) and Karaivanov and Townsend (2014) and. For an empirical test between full insurance versus informational constraints, see Ligon (1998).

Other differences include that our analysis is static while the above papers are inherently dynamic, and in our paper agents perfectly observe local information (but not beyond), while in the above papers incomes are not observable even between the connected pair of agents.
stating that the ratios of marginal utilities of consumptions between any two agents have to be equalized across states. Characterizing the set of Pareto efficient arrangements subject to the local information constraints is technically more challenging, as such constraints are intertwined (different pairs of connected agents are allowed to condition their transfers on different sub-vectors of the state), making it impossible to change transfer arrangements in one state independently of other states. This makes the optimization problem inherently infinite-dimensional. Nevertheless, we can generalize the Borch rule, to a form in conditional expectations. In particular, a necessary and sufficient condition for Pareto optimality of a risk-sharing arrangement with local information constraints states that, for each connected pair of agents, the ratios of expected marginal utilities of consumption conditional on different local states (different realizations of commonly observed endowments), where the conditional expectation is taken over transfers between the paired agents and their non-common neighbors based on non-local information (endowment realizations of non-common neighbors of the pair). As in the context of risk-sharing arrangements with global information, it can be shown that for each set of Pareto-weights\(^5\) there is a unique Pareto efficient consumption plan, characterized by the expectation-form Borch rule (for every realization of local information, for every connected pair of agents).

After the general characterization of Pareto efficient risk-sharing arrangements in terms of first-order conditions, we turn to more explicitly characterizing efficient agreements in a framework with CARA utilities and jointly normally distributed endowments. The characterization is particularly simple for independent endowments: any agent with \(d\) neighbors keeps a \(\frac{1}{d+1}\) fraction of her endowment shock for herself, and transfers a \(\frac{1}{d+1}\) fraction of her endowment shock to each of her neighbors; on top of that, the arrangement can specify state independent transfers, only affecting the distribution of surplus but not the aggregate welfare. This type of transfer arrangement, which we refer to as local equal sharing rule, was considered as an ad hoc sharing rule in \cite{GaoMoon2016}. Our result provides micro-foundations for the rule, in the context of CARA utilities and independent normally distributed endowments. The rule is particularly simple, in that bilateral transfers are linear in endowment realizations, and they only depend on the pair’s endowment realizations, not on those of common friends.

In the case of correlated endowment shocks, even when imposing symmetry across pairwise correlations, providing an analytical solution for general networks is not feasible. Nevertheless, by transforming the system of necessary and sufficient first-order conditions for optimum into a tractable form, we show that there exists a transfer arrangement with the same simple properties as in the case of independent endowments: linear in endowment realizations, and strictly bilateral.

Analyzing how the above optimal arrangement differs from the local equal sharing rule, in the more realistic case of positively correlated endowments, we find that if agents \(i\) and \(j\) are connected, increasing the exposure of \(i\) to transfers from non-common friends increases the share of own endowment shock \(i\) transfers to \(j\), relative to local equal sharing, and it decreases the share of \(j\)’s endowment shock that \(i\) takes from \(j\). These correction terms, which

\(^5\)Just like in the standard setting with risk-sharing arrangements that can be conditioned on global information, for arrangements that can only be conditioned on local information it also holds that the set of Pareto efficient risk-sharing arrangements are equivalent to the set of solutions for a utilitarian planner’s problem, for different weights.
are complicated functions of the network structure, take it into account that more centrally
distributed agents are more exposed to the common shock component, and optimally correct
for them. We define a potential function on the set of agents that measures shock exposure
under the optimal arrangement, and has the property that the agent with the higher potential
transfers a higher fraction of her endowment shock to the other one. Since the potential
function can be computed from the network structure, the latter property is potentially
empirically testable.

Even with the above correction terms relative to local equal sharing, in our model more
central individuals end up with a higher consumption variance. This is because with risk-
sharing contracts that can only be conditioned on local information, these agents become
quasi insurance providers to more peripheral neighbors. For a fixed set of welfare weights,
they get compensated for this service through higher state independent transfers (“insurance
premium”). This contrasts with the predictions from models with enforcement constraints,
like AMS, in which more centrally connected agents are better insured (end up with smaller
consumption variance) because for typical endowment realizations they end up on larger
“risk-sharing islands.” While the centrality implied by the potential function in our model
is not equivalent to more standard notions of centrality, for typical networks the contrast
between the predictions of our model versus the AMS model remains when we use degree
or eigenvector centrality as a centrality measure. We show this through simulations, using
the graph of an Indian village network, obtained from data collected by Field and Pandey.
With simulated endowment shocks, the AMS model produces a negative correlation between
either degree or eigenvalue centrality and consumption variance, more starkly for relatively
tight capacities, while the optimal contracts in our model yield positive correlations.

Our model also provides an explanation why informal insurance might perform badly in
one setting but quite well in another, despite similarities across the settings in average cor-
correlation between endowment realizations and in the network structure. In particular, when
agents can only contract on local information, high correlation between neighboring agents
really hurts risk-sharing efficiency. We demonstrate this using a circle network, assuming
that correlations between endowments geometrically decay with distance: $\rho^k$ for two agents
away from each other. For a large number of agents $n$ and high $\rho$, we show that while
risk-sharing arrangements using global information can achieve almost full efficiency, ar-
rangements using local information can improve very little relative to autarky. We also show
that, for any $n$ and $\rho$, with a symmetric correlation structure (not depending on distance)
that achieves the same variance of consumption under arrangements with global information,
risk sharing is strictly better under the symmetric correlation structure than under the
decaying correlation structure. This might help explain why while in most settings empirical
research found that informal insurance works well, Kazianga and Udry (2006) found a setting
in which informal insurance does not seem to help, and Goldstein, de Janvry, and Sadoulet
(2001) found that certain types of shocks are not well insured through informal risk sharing.

The rest of this paper is organized as follows. In Section 2, we illustrate the difference

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6 We call it potential function because it has analogous properties to the potential function defined in
electronic engineering for electric resistor networks. In particular, it satisfies Kirchoff’s voltage law, in that
the sum of the potential along any cycle in the network is zero. We use this property in proving that there
is an optimal transfer arrangement that is linear in endowment realizations.

7 Insert citation
between risk sharing with global versus local information in the context of a simple example. In Section 3, we derive a set of necessary and sufficient conditions for Pareto efficient transfer arrangements subject to the local information constraints, for general specifications of our model. In Section 4 we more explicitly characterize Pareto efficient transfer arrangements subject to the local information constraints in settings with CARA utilities and jointly normally distributed endowments, and examine properties of these arrangements. In Section 5 we provide further discussions, on how the correlation structure on endowments influences the efficiency of risk sharing with local information relative to the global information benchmark, and on endogenous formation of risk-sharing networks in a context of risk sharing with local information.

2 Example: A 3-Agent Line Network

2.1 Specification

Before investigating general network structures, we consider the simplest example of a non-trivial network, where three agents, denoted by 1, 2 and 3, are minimally connected: agent 1 is linked with agent 2 and agent 3, but agent 2 is not linked with agent 3. Despite its simplicity, this example provides some useful insights on how local information constraints affect efficient risk-sharing agreements.

Assume that agents have homogeneous CARA preferences, of the form $u(x) = -e^{-rx}$. Furthermore, assume that $e_1, e_2, e_3$ are independent and normally distributed, with mean 0 and variance $\sigma^2$. Only linked agents may enter into risk-sharing agreements to insure against endowment risks. Let $t_2$ denote the ex-post transfer from agent 1 to agent 2, $t_3$ the transfer from agent 1 to agent 3. Let $x_1, x_2, x_3$ denote the final allocation of endowments after the transfers are implemented, i.e.,

$$\begin{align*}
    x_1 &= e_1 - t_2 - t_3 \\
    x_2 &= e_2 + t_2 \\
    x_3 &= e_3 + t_3
\end{align*}$$

Below we compare Pareto efficient transfer rules in two cases: under agreements between neighboring agents that can condition transfers on everyone’s endowment realization (global
information), and under agreements that can only condition on endowment realizations of the two agents forming the agreement (local information).

2.2 Agreements with Global Information

First we consider the benchmark case when bilateral transfer agreements between neighboring agents can be conditioned on global information, that is on all three agents’ endowment realizations, so that $t_2, t_3$ can be arbitrary functions of the endowments $e_1, e_2, e_3$. Standard arguments (see Wilson (1968)) establish that Pareto efficient transfer rules $t_2, t_3$ are the ones maximizing the social planner’s problem:

$$
\mathbb{E} \left[ \sum_{i=1}^{3} \lambda_i u (x_i) \right] = \mathbb{E} \left[ \lambda_1 u (e_1 - t_2 - t_3) + \lambda_2 u (e_2 + t_2) + \lambda_3 u (e_3 + t_3) \right],
$$

for some $\lambda_1, \lambda_2, \lambda_3 \in [0, 1]$ s.t. $\lambda_1 + \lambda_2 + \lambda_3 = 1$. By the well-known Borch rule (Borch (1962), Wilson (1968)), the necessary and sufficient conditions for optimum are given by:

$$
\lambda_1 u' (e_1 - t_2 - t_3) = \lambda_2 u' (e_2 + t_2) = \lambda_3 u' (e_3 + t_3) \quad \forall e_1, e_2, e_3.
$$

With our CARA utility specification, this yields

$$
\begin{align*}
t_2 (e_1, e_2, e_3) &= \frac{1}{3} e_1 - \frac{2}{3} e_2 + \frac{1}{3} e_3 - \frac{1}{3} r \ln (\lambda_2 \lambda_3 / \lambda_1^2) \\
t_3 (e_1, e_2, e_3) &= \frac{1}{3} e_1 + \frac{2}{3} e_2 - \frac{2}{3} e_3 - \frac{1}{3} r \ln (\lambda_1 \lambda_2 / \lambda_3^2)
\end{align*}
$$

and the implied final allocations for any $(e_1, e_2, e_3)$ are:

$$
\begin{align*}
x_1 &= \frac{1}{3} (e_1 + e_2 + e_3) + \frac{1}{3} r \ln (\lambda_2 \lambda_3 / \lambda_1^2) \\
x_2 &= \frac{1}{3} (e_1 + e_2 + e_3) + \frac{1}{3} r \ln (\lambda_1 \lambda_3 / \lambda_2^2) \\
x_3 &= \frac{1}{3} (e_1 + e_2 + e_3) + \frac{1}{3} r \ln (\lambda_1 \lambda_2 / \lambda_3^2)
\end{align*}
$$

(2)

That is, Pareto efficient transfer agreements in every state divide total realized endowment equally among agents, and the equal division is then corrected by state-independent transfers that achieve the welfare weights.

2.3 Agreements with Local information

Suppose now that endowment realizations are only locally observable and verifiable, so that transfers agreements $t_2, t_3$ can be contingent on local information only, that is:

$$
t_2 = t_2 (e_1, e_2), \quad t_3 = t_3 (e_1, e_3).
$$

First, note that the set of state contingent consumption plans that can be achieved by such transfer agreements is still convex, since if consumption plan $x$ is achieved by transfer arrangements $(t_2, t_3)$ and consumption plan $x'$ is achieved by transfer arrangements $(t'_2, t'_3)$ then consumption plan $\alpha x + (1 - \alpha) x$ is achieved by transfer arrangements

...
The necessary and sufficient FOC for this maximization problem is:

\[
\text{distribution of realization of condition for a transfer arrangement to be optimal for the planner is that, for any given when transfer agreements can only condition on local information. However, a necessary welfare function (1) for some welfare weights, even when the transfers can only be conditioned on local information.}
\]

Second, achieving consumption plans on the Pareto frontier, given by (2), is not possible when transfer agreements can only condition on local information. However, a necessary condition for a transfer arrangement to be optimal for the planner is that, for any given realization of \( e_1 \) and \( e_2 \), \( t_2 \) should maximize \( \lambda_1 u (e_1 - t_2 - t_3) + \lambda_2 u (e_2 + t_2) \), given the distribution of \( e_3 \) conditional on \( e_1 \) and \( e_2 \), and the implied distribution of consumption levels (net of \( t_2 \)) induced by \( t_3 \). In short, given \( t_3, t_2 \) should maximize the planner’s welfare function:

\[
\max_{t_2} \int [\lambda_1 u (e_1 - t_2 - t_3) + \lambda_2 u (e_2 + t_2)] f_{3|12} (e_3) \, de_3
\]

The necessary and sufficient FOC for this maximization problem is:

\[
\lambda_1 E \left[ u' (e_1 - t_2 - t_3 (e_1, e_3)) \mid e_1, e_2 \right] = \lambda_2 u' (e_2 + t_2).
\]

Similarly, the FOC for \( t_3 \) to be optimal, given \( t_2 \) is:

\[
\lambda_1 E \left[ u' (e_1 - t_2 (e_1, e_2) - t_3) \mid e_1, e_2 \right] = \lambda_3 u' (e_3 + t_3)
\]

The above arguments establish that any Pareto transfer arrangement has to satisfy both (3) and (4), so efficient \( t_2, t_3 \) are the solutions for the system of integral equations.

Then, with the CARA utility specification:

\[
\lambda_1 E \left[ r e^{-r(e_1-t_2-t_3 \mid e_1, e_2)} \mid e_1, e_2 \right] = \lambda_2 r e^{-r(e_2+t_2)}
\]

\[
\Leftrightarrow t_2 = \frac{1}{2} e_1 - \frac{1}{2} e_2 - \frac{1}{2} \ln (E \left[ e^{rt_3 \mid e_1, e_3} \mid e_1 \right]) - \frac{1}{2r} \ln (\lambda_1/\lambda_2).
\]

Defining \( T_2 := E \left[ e^{rt_2 \mid e_1, e_2} \mid e_1 \right] \) and \( T_3 := E \left[ e^{rt_3 \mid e_1, e_3} \mid e_1 \right] \), we get:

\[
T_2 = E \left[ e^{rt_2 \mid e_1, e_2} \mid e_1 \right] = e^{r \left( \frac{1}{2} e_1 - \frac{1}{2} e_2 - \frac{1}{2} \ln (\lambda_1/\lambda_2) \right)} \cdot \int e^{-\frac{1}{2} e_2} \cdot \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{e_2^2}{2\sigma^2}} \, de_2
\]

\[
= e^{r \left( \frac{1}{2} e_1 - \frac{1}{2} \ln (\lambda_1/\lambda_2) \right)} \cdot \int e^{-\frac{1}{2} e_2} \cdot \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{e_2^2}{2\sigma^2}} \, de_2
\]

With a similar equation for \( T_3 \), we can solve for \((T_2, T_3)\):

\[
\begin{align*}
T_2 &= e^{r \left( \frac{1}{2} e_1 + \frac{1}{2} \sigma^2 - \frac{1}{2} \ln (\lambda_1/\lambda_2^2) \right)} \\
T_3 &= e^{r \left( \frac{1}{2} e_1 + \frac{1}{2} \sigma^2 - \frac{1}{2} \ln (\lambda_1/\lambda_2^2) \right)}
\end{align*}
\]

Substituting \((T_2, T_3)\) into the expressions for \( t_2, t_3 \) above, we get the Pareto efficient transfer agreements:

\[
\begin{align*}
t_2 (e_1, e_2) &= \frac{1}{3} e_1 - \frac{1}{2} e_2 - \frac{1}{24} r \sigma^2 - \frac{1}{36} \ln (\lambda_1 \lambda_3 / \lambda_2^2) \\
t_3 (e_1, e_3) &= \frac{1}{3} e_1 - \frac{1}{2} e_2 - \frac{1}{24} r \sigma^2 - \frac{1}{36} \ln (\lambda_1 \lambda_2 / \lambda_3^2)
\end{align*}
\]
Notice the transfer rules can be decomposed into three parts. The first part, \( \frac{1}{3}e_1 - \frac{1}{2}e_2 \), corresponds to the “local equal sharing rule”, which is the local variant of the equal sharing rule. It implies that individual \( i \)’s endowment \( e_i \) is equally shared by \( i \) and \( i \)’s neighbors, i.e.,

\[
t_{ij} = \frac{e_i}{d_i + 1} - \frac{e_j}{d_j + 1}.
\]

The second part corresponds to a state-independent transfer that can be regarded as the “insurance premium” paid by the “net insurance purchaser” to the “net insurance provider”. In this case, as the final wealth levels are

\[
\begin{align*}
x_1 &= \frac{1}{3}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 + \frac{1}{12}r\sigma^2 + \frac{1}{3\rho} \ln \left( \frac{\lambda_1^2}{\lambda_2^2\lambda_3} \right), \\
x_2 &= \frac{1}{3}e_1 + \frac{1}{2}e_2 - \frac{1}{24}r\sigma^2 + \frac{1}{3\rho} \ln \left( \frac{\lambda_2^2}{\lambda_1^2\lambda_3} \right), \\
x_3 &= \frac{1}{3}e_1 + \frac{1}{2}e_3 - \frac{1}{24}r\sigma^2 + \frac{1}{3\rho} \ln \left( \frac{\lambda_3^2}{\lambda_1^2\lambda_2} \right),
\end{align*}
\]

agent 1 undertakes extra risk exposure \( \frac{1}{3}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 \) in comparison to the exposures undertaken by agent 2 or 3, \( \frac{1}{3}e_1 + \frac{1}{2}e_2 \) or \( \frac{1}{3}e_1 + \frac{1}{2}e_3 \). Hence, agent 1 is rewarded the certainty equivalent (CE) for her intermediary role in the risk sharing. The third part redistributes wealth according to the weights put on different agents (it is 0 when \( \lambda_1 = \lambda_2 = \lambda_3 \)).

To evaluate the welfare loss associated with agreements to be conditional only on local information, note that because social welfare is a linear, strictly decreasing function of total variances under CARA utilities, we can simply compare the total variances of final allocations.

With agreements conditional on global information, the sum of consumption variances is:

\[
TVar_G = 3 \cdot Var \left[ \frac{e_1 + e_2 + e_3}{3} \right] = \sigma^2.
\]

With bilateral agreements subject to the local information constraints, the sum of consumption variances increases to:

\[
TVar_L = Var \left[ \frac{1}{3}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 \right] + Var \left[ \frac{1}{3}e_1 + \frac{1}{2}e_2 \right] + Var \left[ \frac{1}{3}e_3 + \frac{1}{2}e_2 \right] = \frac{4}{3} \sigma^2.
\]

Hence the welfare loss arising from constraining agreements to be conditional on local information is \( \frac{1}{3} \sigma^2 \) in the above example.

### 3 General Conditions for Pareto Efficiency

Before we proceed to our main analysis, we introduce some notations. Let \( N = \{1, 2, ..., n\} \) be a finite set of agents and let \( G \) be the matrix representing the network structure of \( N \). A pair of agents \( i, j \) are linked if \( G_{ij} = 1 \), and by convention, \( G_{ii} = 0 \). Throughout the paper we assume, without loss of generality, that \( G \) represents a connected network\(^8\)

\(^8\)Otherwise we may analyze each component separately.
Define the neighborhood of $i$ $N_i := \{j \in N : G_{ij} = 1\}$ and the extended neighborhood of $i$ $\overline{N_i} := N_i \cup \{i\}$. The degree of agent $i$ is defined as $d_i := \# (N_i)$, the number of agents to whom $i$ is linked to. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we model each agent’s endowment as a random variable $e_i$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and we denote the joint distribution of the vector of endowments $e \equiv (e_i)_{i \in N}$ by $P$. We use $\mathbb{E}[\cdot]$ to denote the expectation operator under the probability measure $\mathbb{P}$. Furthermore, we sometimes abuse notations and write $\omega \equiv e(\omega) \equiv (e_1(\omega), e_2(\omega), \ldots, e_n(\omega))$, i.e. we treat the state of world as interchangeable with the joint realization of endowments, taking $\mathcal{F} = \mathcal{B}(\mathbb{R}^n)$.

A central assumption in our paper is that agents can only observe the endowment realizations of their direct neighbors. Define $N_{ij} := N_i \cap N_j$ and $\overline{N_{ij}} := \overline{N_i} \cap \overline{N_j}$. Let $I_i = (e_j)_{j \in \overline{N_i}}$, the information vector of $i$. Then the common information of agents $i$ and $j$ is $I_{ij} := (e_k)_{k \in \overline{N_{ij}}}$. We assume that only linked pairs of agents can engage in informal risk sharing directly. Risk sharing takes the form of ex ante agreements between linked agents $i$ and $j$ on a net transfer $t_{ij}$ from $i$ to $j$, conditional on the realization of the state. We assume that agents can only condition the transfer on their common information. Formally, we require that $t_{ij} : \Omega \to \mathbb{R}$ be $\sigma(I_{ij})$-measurable, where $\sigma(I_{ij})$ denotes the $\sigma$-algebra induced by $I_{ij}$. By definition, $t_{ij}(\omega) = -t_{ji}(\omega)$ for every $\omega \in \Omega$ and linked $i, j \in N$. We refer to the collection of ex ante risk-sharing agreements of the above form between all pairs of linked agents as a transfer arrangement.

Let $\mathcal{T}$ denote the set of all possible transfer arrangements $t : \Omega \equiv \mathbb{R}^n \to \mathbb{R}^{\sum_{i \in N} d_i}$ that are only contingent on the commonly known information for each linked pair:

$$\mathcal{T} := \left\{ t : \Omega \to \mathbb{R}^{\sum_{i \in N} d_i} \mid \forall i, j \text{ s.t. } G_{ij} = 1, \ t_{ij} \text{ is } \sigma(I_{ij}) \text{-measurable, } \wedge \ t_{ij}(\omega) + t_{ji}(\omega) = 0 \ \forall \omega \right\}.$$  

Define $\mathcal{T}^* \subseteq \mathcal{T}$ by

$$\mathcal{T}^* := \{ t \in \mathcal{T} \mid \langle t, t \rangle < \infty \}$$

where, $\forall s, t \in \mathcal{T}$,

$$\langle s, t \rangle := \mathbb{E} \left[ \sum_{G_{ij} = 1} s_{ij}(\omega) t_{ij}(\omega) \right].$$

It follows that $\langle \cdot, \cdot \rangle$ is an inner product and $\mathcal{T}^*$ is a well-defined Hilbert space (see Lemma 1 in the Appendix A for a formal proof).

It should be pointed out that, rigorously each element in $\mathcal{T}^*$ corresponds to an “equivalent class” of transfer agreements that are indistinguishable under the norm induced by $\langle \cdot, \cdot \rangle$. Moreover, as will be clear in the next three paragraphs, measure-0 changes in the transfers do not change at all our objective functions (5) and (6) in the form of (ex ante or conditional) expected utilities. Hence, throughout the paper, we write “$s = t$” to mean “$\langle s - t, s - t \rangle = 0$”, or equivalently “$s(\omega) = t(\omega)$ a.s. ($\mathbb{P}$)”, and make statements with the understanding that the “a.s. ($\mathbb{P}$)” qualifier applies whenever necessary.

To characterize the set of Pareto efficient transfers under the local information constraint, we solve the following problem:

$$\max_{t \in \mathcal{T}^*} \mathbb{E} \left[ \sum_{k \in N} \lambda_k u_k \left( e_k - \sum_{h \in N_k} t_{kh} \right) \right]$$  

(5)
Since the transfer rule \( t_{ij} \) is restricted to be measurable with respect to \( \sigma (I_{ij}) \), we may, with an abuse of notation, write it as \( t_{ij} : \mathbb{R}^{\text{dim}(I_{ij})} \to \mathbb{R} \). The following proposition provides a formal characterization of the solution to the maximization problem above.

**Proposition 1.** A profile of \( t \in T^* \) solves \([5]\) if and only if, for almost all possible states of world \( \omega \in \Omega \), it simultaneously solves the \( \sum_{i \in N} d_i \) optimization problems in the form of \([6]\) at each common information set of the linked pair \( I_{ij} \):

\[
t_{ij} (I_{ij}) \in \operatorname{arg max}_{t_{ij} \in \mathbb{R}} \mathbb{E} \left[ \lambda_i u_i \left( e_i - \tilde{t}_{ij} - \sum_{h \in N_i \setminus \{j\}} t_{ih} \right) + \lambda_j u_j \left( e_j + \tilde{t}_{ij} - \sum_{h \in N_j \setminus \{i\}} t_{jh} \right) + \sum_{k \neq i, j} \lambda_k u_k \left( e_k - \sum_{h \in N_k} t_{kh} \right) \right]_{I_{ij}} \forall i, j \text{ s.t. } G_{ij} = 1 \text{ a.s.} (\mathbb{P})
\]

**Proof.** See the Appendix. \( \square \)

Proposition 1 is an intuitive result, and the analogue of it is easy to show when transfer agreements can be conditional on everyone’s income realization, as in Wilson (1968). In that case, for each possible realizations of \( \omega \equiv (e_k)_{k \in N} \), we may freely choose \( (t_{ij}(\omega))_{G_{ij}=1} \), a finite dimensional vector, to maximize \([5]\). This enables for standard finite dimensional optimization techniques, leading to first order conditions for optimum that just connect ratios of marginal utilities of consumption for two agents in two different states. In contrast, the intertwined local information structure induced by the locality constraint makes the optimization problem fundamentally infinite-dimensional. State-by-state optimization is no longer feasible: taking for example again the network depicted in Figure 1, we see that the local information set for the pair 12 is defined by \( \{ \tilde{\omega} \in \Omega : e_1 (\tilde{\omega}) = e_1 (\omega), e_2 (\tilde{\omega}) = e_2 (\omega) \} \) for a given \( \omega \), and the transfer \( t_{12} \) must be constant over all values of \( \omega \) on \( I_{12} \). As the value of \( t_{13} \) can vary continuously as a function of \( e_3 \) for each \( \omega \in I_{12} \), the efficient \( t_{13} \) must be optimal in expectation with respect to the distribution of \( t_{13} \) conditional on \( I_{12} \). Similarly, \( t_{13} \) must be optimal with respect to the conditional distribution of \( t_{12} \) on each \( I_{13} \). As the realizations of \( I_{12} \) and the realizations of \( I_{13} \) induce two different uncountable partitions of the state space, with the transfers being constant on respective cells of the partitions, we can no longer carry out state-by-state optimization at each \( \omega \), but have to optimize the transfers at all states simultaneously.

The proof of Proposition 1 formally establishes this using mathematical results from infinite-dimensional convex optimization. We begin by establishing that under the inner product defined above, \( T^* \) forms a Hilbert space. We then show that the objective function is concave and twice Fréchet-differentiable. Then, the sufficient and necessary condition for optimality is given by the first Fréchet-derivative of the objective being the zero function, which is the infinite-dimensional generalization of the usual first order condition for optimality.

Proposition 1 can also be regarded as a decentralization result for the social optimization problem. Notice that in problem \([6]\), at each \( I_{ij} \), the choice of \( t_{ij} \) affects the expected utilities of only \( i \) and \( j \), so each optimization problem in \([6]\) can be reinterpreted as the surplus maximization problem jointly solved by the linked pair \( ij \), given the transfer rules...
chosen by other linked pairs. Then, the system of optimizations solved by all pairs in (6) can be regarded as a notion of equilibrium: each linked pair optimally chooses the transfer agreement along this link, given other transfer agreements along other links. Hence, Pareto efficiency on the social level is achieved whenever each pair makes efficient (defined for this pair only) choice on transfer rule.

The next result establishes that while in general there can be multiple transfer arrangements satisfying the conditions for optimality (6), they all imply the same consumption plan in all states.

**Proposition 2.** All profiles of transfer rules $t \in T^*$ that solve (6) lead to ($\mathbb{P}$-almost) the same profile of final allocations $x$, where

$$x_i := e_i - \sum_{j \in N_i} t_{ij}.$$  

**Proof.** See the Appendix.

By Proposition 2, if we can find a profile of transfer arrangements so that the induced allocation rules satisfy (6), then it must be a Pareto efficient profile of transfer arrangements.

For simplicity, below we will denote the conditional expectation operator $\mathbb{E} [\cdot | I_{ij}]$ by $E_{ij} [\cdot]$. In observation of Proposition 1 and Proposition 2, we may express the necessary and sufficient condition for Pareto efficiency as a requirement on the ratio of conditional expected marginal utilities given by the next Corollary.

**Corollary 1.** A profile of transfer arrangements $t$ is Pareto efficient if and only if the ratio of the expected marginal utilities conditional on all possible common information sets is constant:

$$\frac{E_{ij} [u'_i(x_i)]}{E_{ij} [u'_j(x_j)]} = \frac{\lambda_j}{\lambda_i}.$$  

holds for every $I_{ij} \in I_{ij}$ and every $i, j \in N$ s.t. $G_{ij} = 1$.

**Proof.** By the concavity (See Lemma 2 in the Appendix for a rigorous proof) of the objective function in (6), the FOC is both sufficient and necessary for maximization. Taking FOC w.r.t $t_{ij}$, we have

$$\mathbb{E} \left[ \lambda_i u'_i \left( e_i - \sum_{h \in N_i} t_{ih} \right) + \lambda_j u'_j \left( e_j - \sum_{h \in N_j} t_{jh} \right) \cdot (-1) \bigg| I_{ij} \right] = 0$$

Rearranging the above we have

$$\frac{E_{ij} [u'_i(x_i)]}{E_{ij} [u'_j(x_j)]} = \frac{\mathbb{E} \left[ u'_i \left( e_i - \sum_{h \in N_i} t_{ih} \right) \bigg| I_{ij} \right]}{\mathbb{E} \left[ u'_j \left( e_j - \sum_{h \in N_j} t_{jh} \right) \bigg| I_{ij} \right]} = \frac{\lambda_j}{\lambda_i}.$$  

\[\square\]
This extends the Borch rule (Borch (1962), Wilson (1968)) for Pareto efficient risk-sharing agreements to settings with local information constraints. As opposed to the case when transfer agreements can be conditioned on the endowment realizations of all players, the ratio of expected marginal utilities of consumptions among agents do not have to be equal state by state, they only have to be equal between linked agents in expectation, conditional on common information.

4 Efficient Agreements under the CARA-Normal Setting

In this section we investigate Pareto efficient risk-sharing arrangements, subject to local information constraints, under the assumption of CARA utilities and jointly normally distributed endowments.

**Assumption 1.** Throughout the subsequent sections we assume that agents have homogeneous CARA utility functions \( u(x) = -e^{-rx} \), where \( r > 0 \) is the coefficient of absolute risk aversion. The vector of endowments \( (e_i)_{i \in N} \) follows a multivariate normal distribution, \( e \sim N(0, \sigma^2 \Sigma) \).

4.1 Independent Endowment Shocks

We first analyze the case where endowment shocks are independently distributed, i.e., \( e \sim N(0, \sigma^2 \cdot I_n) \).

We use a guess and verify approach, and postulate that a linear transfer scheme, that is a scheme for which the transfer between any two connected agents is a linear function of endowment realizations in the pair’s joint information set, can achieve any Pareto efficient risk-sharing arrangement.

Given a linear transfer scheme, the final consumption levels, conditional on \( I_{ij} \), also follow normal distribution. Hence,

\[
E_{ij} \left[ u'_i (x_i) \right] = r E_{ij} \left[ e^{-rx_i} \right] = r e^{-r (E_{ij}[x_i] - \frac{1}{2} r Var_{ij} [x_i])}.
\]

Define

\[
CE (x_i | I_{ij}) := E_{ij} [x_i] - \frac{1}{2} r Var_{ij} [x_i],
\]

and (7) can then be rewritten as

\[
\exp \left[ -r \left( CE (x_i | I_{ij}) - CE (x_j | I_{ij}) \right) \right] = \frac{\lambda_j}{\lambda_i}
\]

\( \Leftrightarrow \)

\[
CE (x_i^* | I_{ij}) - \frac{1}{r} \ln \lambda_i = CE (x_j^* | I_{ij}) - \frac{1}{r} \ln \lambda_j
\] (8)
The profile of transfer schemes \( t \) achieves Pareto efficiency if and only if (8) holds for every pair of \( ij \) s.t. \( G_{ij} = 1 \), i.e., the difference in the conditional certainty equivalents is constant at each intersection of the information sets of a linked pair.

We say a profile of transfer rules is strictly bilateral if \( t_{ij} (\omega) = t_{ij} (e_i, e_j) \).

**Proposition 3.** Given any profile of positive welfare weights \((\lambda_i)_{i \in N}\), there always exists a strictly bilateral Pareto efficient profile of transfer rules in \( T^* \) in the form of

\[
t_{ij}^* (e_i, e_j) := \frac{e_i}{d_i + 1} - \frac{e_j}{d_j + 1} + \mu_{ij}^*
\]

for some \( \mu_{ij}^* \in \mathbb{R} \), for each linked pair \( ij \).

**Proof.** Let \( x_i^* \) be the final consumption induced by the transfer \( t^* \) described above. Then

\[
CE (x_i^* | I_{ij}) = E_{ij} \left[ e_i - \sum_{k \in N_i} t_{ik}^* \right] - \frac{1}{2} r Var_{ij} \left[ e_i - \sum_{k \in N_i} t_{ik}^* \right] = \frac{e_i}{d_i + 1} - \frac{e_j}{d_j + 1} - \frac{1}{2} r Var_{ij} \left[ \sum_{k \in N_i} \frac{e_k}{d_k + 1} \right] + \frac{1}{2} r \sigma^2 \cdot \sum_{k \in N_i \setminus N_j} \frac{1}{(d_k + 1)^2}.
\]

The necessary and sufficient condition for \( t^* \) to be Pareto efficient is given by (8). Plugging the above into (8) and canceling out the terms dependent on local information \((e_k)_{k \in N_i \setminus N_j}\), we arrive at the following condition for Pareto efficiency:

\[
\sum_{k \in N_i} \mu_{ik}^* + \frac{1}{2} r \sigma^2 \cdot \sum_{k \in N_i \setminus N_j} \frac{1}{(d_k + 1)^2} = \frac{1}{r} \ln \lambda_i - \frac{1}{r} \ln \lambda_j.
\]

Any profile of state-independent transfers \( \mu^* \) that solves the above system (9) makes \( t^* \) efficient under weightings \( \lambda \).

Notice that, if \( CE (x_i^* | I_{ij}) - \frac{1}{r} \ln \lambda_i = CE (x_j^* | I_{ij}) - \frac{1}{r} \ln \lambda_j \) holds for any \( \omega \), then

\[
CE (x_i^*) - \frac{1}{r} \ln \lambda_i = E \left[ CE (x_i^* | I_{ij}) - \frac{1}{r} \ln \lambda_j \right] - Var \left[ CE (x_i^* | I_{ij}) - \frac{1}{r} \ln \lambda_i \right] = CE (x_j^*) - \frac{1}{r} \ln \lambda_j
\]

Hence, with \( G \) assumed WLOG to be connected, we have

\[
CE (x_i^*) - \frac{1}{r} \ln \lambda_i = \frac{1}{n} \sum_{k \in N} \left( CE (x_k^*) - \frac{1}{r} \ln \lambda_k \right)
\]
\[ r\sigma^2 - \frac{1}{2n} \sum_{k \in N} \frac{1}{d_k + 1} - \frac{1}{nr} \sum_{k \in N} \ln \lambda_k \]  
(10)

On the other hand, as \( x^*_i = e_i + \sum_{k \in N_i} \left( \frac{e_k}{d_k + 1} - \mu^*_ik \right) \),

\[ CE(x^*_i) = -\sum_{k \in N_i} \mu^*_ik - \frac{1}{2} r\sigma^2 \sum_{k \in N_i} \frac{1}{(d_k + 1)^2} \]  
(11)

Equating the expressions for \( CE(x^*_i) \) in (10) and (11), we obtain

\[ \sum_{k \in N_i} \mu^*_ik = \frac{1}{2} r\sigma^2 \left( \frac{1}{n} \sum_{k \in N} \frac{1}{d_k + 1} - \sum_{k \in N_i} \frac{1}{(d_k + 1)^2} \right) + \frac{1}{r} \left( \frac{1}{n} \sum_{k \in N} \ln \lambda_k - \ln \lambda_i \right) \]  
(12)

Lemma 6 in the Appendix establishes that there indeed exists a solution \( \mu^* \) to (12). Given any solution \( \mu^* \) to (12), as \( N_i \setminus (N_i \setminus N_j) = N_{ij} \), we have

\[ \sum_{k \in N_i} \mu^*_ik + \frac{1}{2} r\sigma^2 \sum_{k \in N_i \setminus N_j} \frac{1}{(d_k + 1)^2} + \frac{1}{r} \ln \lambda_i = \frac{1}{2} r\sigma^2 \left( \frac{1}{n} \sum_{k \in N} \frac{1}{d_k + 1} - \sum_{k \in N_{ij}} \frac{1}{(d_k + 1)^2} \right) + \frac{1}{nr} \sum_{k \in N} \ln \lambda_k \]

\[ = \sum_{k \in N_j} \mu^*_jk + \frac{1}{2} r\sigma^2 \sum_{k \in N_j \setminus N_i} \frac{1}{(d_k + 1)^2} + \frac{1}{r} \ln \lambda_j \]

implying that \( \mu^* \) also solves the system of equations (9). Hence, \( t^* \) is Pareto efficient.  

Recall that, by Proposition 2, the Pareto efficient profile of final consumptions is always unique. But for general networks, there might be multiple transfer arrangements that are Pareto efficient. In particular, superfluous transfers, which can be either state-dependent or state-independent, may be freely added to a cycle of individuals in the network without changing the final consumptions. Therefore, in general the transfer scheme achieving a Pareto efficient risk-sharing arrangement is not unique. In Appendix B we show that for tree networks the linear transfer scheme featured in Proposition 3 is the unique transfer scheme that achieves a given Pareto efficient risk-sharing arrangement.

The efficient transfer \( t^*_{ij}(e_i, e_j) \) subject to the locality constraint is composed of two parts: the state-contingent “local equal sharing rule” and the state-independent “insurance premium”. Furthermore, the transfers between two connected agents ascribed by the linear transfer scheme in Proposition 3 only depend on endowment realizations of the two of them, not of their common neighbors. That is, only bilateral information is required for efficient risk sharing with local information.

\[ ^9 \text{In Lemma 6, we establish the existence of solutions for a more general system of equations, which will also be useful elsewhere.} \]
4.2 Correlated Endowment Shocks

We now turn to the case of correlated endowment shocks. To maintain analytical tractability, we assume a symmetric correlation structure, where any two individuals’ endowments have a constant pairwise correlation coefficient of \( \rho \in (-\frac{1}{n-1}, 1] \). Formally, we assume that

\[
e \sim N \left(0, \sigma^2 \cdot \Sigma \right)
\]

where

\[
\Sigma := \begin{pmatrix}
1 & \rho & \cdots & \rho \\
\rho & 1 & \cdots & \rho \\
\vdots & \vdots & \ddots & \vdots \\
\rho & \rho & \cdots & 1
\end{pmatrix},
\]

Equivalently, we are assuming that each individual’s endowment shock can be decomposed additively into two independent components:

\[
e_i = \sqrt{\rho} \tilde{e}_0 + \sqrt{1-\rho} \tilde{e}_i,
\]

with \( (\tilde{e}_k)_{k=0}^n \sim iid \ N \left(0, \sigma^2 \right) \).

We will use the following mathematical result below on the conditional distribution of a sub-vector of \( e \):

\[
e_{>k} \mid e_{\leq k} \sim N \left( \Sigma_{[n-k,k]} \Sigma_{[k]}^{-1} e_{\leq k}, \Sigma_{(n-k,k)} - \Sigma_{[n-k,k]} \Sigma_{[k]}^{-1} \Sigma'_{[n-k,k]} \right).
\]

where

\[
\Sigma_{(k)} := \begin{pmatrix}
1 & \rho & \cdots & \rho \\
\rho & 1 & \cdots & \rho \\
\vdots & \vdots & \ddots & \vdots \\
\rho & \rho & \cdots & 1
\end{pmatrix}_{k \times k}, \quad \Sigma_{[h,k]} := \begin{pmatrix}
\rho & \rho & \cdots & \rho \\
\rho & \rho & \cdots & \rho \\
\vdots & \vdots & \ddots & \vdots \\
\rho & \rho & \cdots & 1
\end{pmatrix}_{h \times k}.
\]

Below we show that any Pareto efficient risk-sharing arrangement can be achieved through a linear transfer scheme.

We first consider the case of minimally connected networks, for simplicity of notations, and in order to develop intuition. Notice that, under minimal connectedness, \( I_{ij} = (e_i, e_j) \), so transfer \( t_{ij} \) must be strictly bilateral. Then, the local FOC for optimality can be written as

\[
E \left[ r e^{-r(e_i - t_{ij} - \sum_{k \in N_i \setminus \{j\}} t_{ik}(e_i, e_k))} \mid e_i, e_j \right] = \alpha_{ij} E \left[ r e^{-r(e_j + t_{ij} + \sum_{h \in N_j \setminus \{i\}} t_{jh}(e_j, e_h))} \mid e_i, e_j \right]
\]

or equivalently

\[
t_{ij} = \frac{1}{2} e_i - \frac{1}{2} e_j - \frac{1}{2r} \ln E \left[ e^{r \sum_{k \in N_i \setminus \{j\}} t_{ik}(e_i, e_k)} \mid e_i, e_j \right] + \frac{1}{2r} \ln E \left[ e^{r \sum_{h \in N_j \setminus \{i\}} t_{jh}(e_j, e_h)} \mid e_i, e_j \right]
\]

\[(13)\]

\( \frac{10}{n-1} - \frac{1}{n-1} \) is the lower bound for a global pairwise correlation in a \( n \)-person economy; mathematically, it is the smallest \( \rho \) such that the variance-covariance matrix is positive semi-definite. For any \( \rho > -\frac{1}{n-1} \), the variance-covariance matrix is positive definite.

\( \text{See Eaton (2007), p.116-117.} \)
Postulating a linear transfer scheme of the form:

\[ t_{ij}(e_i, e_j) = \alpha_{ij}e_i - \alpha_{ji}e_j + \mu_{ij} \quad \forall G_{ij} = 1, \]

we can write

\[
\sum_{k \in N \setminus \{j\}} t_{ik}(e_i, e_k) = e_i \sum_{k \in N \setminus \{j\}} \alpha_{ik} - \sum_{k \in N \setminus \{j\}} \alpha_{ki}e_k + \sum_{k \in N \setminus \{j\}} \mu_{ik}
\]

and similar for \( \sum_{k \in N \setminus \{i\}} t_{jk}(e_j, e_k) \). Plugging the above into (13) yields:

\[
t_{ij} = \frac{1}{2}e_i - \frac{1}{2}e_j - \frac{1}{2} \sum_{k \in N \setminus \{j\}} \alpha_{ik}e_i - \frac{1}{2r} \ln \mathbb{E} \left[ e^{-r \sum_{k \in N \setminus \{j\}} \alpha_{ki}e_k} \middle| e_i, e_j \right] + \frac{1}{2} \sum_{k \in N \setminus \{i\}} \alpha_{jk}e_j + \frac{1}{2r} \ln \mathbb{E} \left[ e^{-r \sum_{k \in N \setminus \{i\}} \alpha_{kj}e_k} \middle| e_i, e_j \right] + \mu_{ij}
\]

Note that the distribution of \( e_k \) conditional on \( (e_i, e_j) \) is

\[
e_k \middle| e_i, e_j \sim N \left( \frac{\rho}{1 + \rho} (e_i + e_j), \frac{1 + \rho - 2\rho^2}{1 + \rho} \cdot \sigma^2 \right)
\]

and the correlation between \( e_h \) and \( e_k \) conditional on \( (e_i, e_j) \) is \( \frac{\rho}{1 + 2\rho} \). Therefore

\[
\sum_{k \in N \setminus \{j\}} \alpha_{ki}e_k \sim N \left( \frac{\rho}{1 + \rho} (e_i + e_j) \sum_{k \in N \setminus \{j\}} \alpha_{ki}, V_{i \setminus j} \right)
\]

where

\[
V_{i \setminus j} := \frac{1 + \rho - 2\rho^2}{1 + \rho} \cdot \sigma^2 \cdot \left( \sum_{k \in N \setminus \{j\}} \alpha_{ki}^2 + \frac{2\rho}{1 + 2\rho} \sum_{h < k \in N \setminus \{j\}} \alpha_{hi} \alpha_{ki} \right)
\]

Hence,

\[
t_{ij} = \frac{1}{2}e_i - \frac{1}{2}e_j - \frac{1}{2} \sum_{k \in N \setminus \{j\}} \alpha_{ik}e_i + \frac{1}{2} \sum_{k \in N \setminus \{i\}} \alpha_{jk}e_j
\]

\[
+ \frac{1}{2} \left( \frac{\rho}{1 + \rho} (e_i + e_j) \sum_{k \in N \setminus \{j\}} \alpha_{ki} - \frac{1}{2} r \sigma^2 \cdot V_{i \setminus j} \right)
\]

\[
- \frac{1}{2} \left( \frac{\rho}{1 + \rho} (e_i + e_j) \sum_{k \in N \setminus \{i\}} \alpha_{kj} - \frac{1}{2} r \sigma^2 \cdot V_{j \setminus i} \right)
\]

\[
= \frac{1}{2} \left[ 1 - \sum_{k \in N \setminus \{j\}} \alpha_{ik} + \frac{\rho}{1 + \rho} \left( \sum_{k \in N \setminus \{j\}} \alpha_{ki} - \sum_{k \in N \setminus \{i\}} \alpha_{kj} \right) \right] e_i
\]
- \frac{1}{2} \left[ 1 - \sum_{k \in N_i \setminus \{i\}} \alpha_{jk} + \frac{\rho}{1+\rho} \left( \sum_{k \in N_i \setminus \{i\}} \alpha_{kj} - \sum_{k \in N_i \setminus \{j\}} \alpha_{ki} \right) \right] e_j + C_{ij}

where \( C_{ij} \) is some constant. Reconciling the above with the postulation that \( t_{ij}(e_i, e_j) = \alpha_{ij} e_i - \alpha_{ji} e_j + \mu_{ij} \), we have

\[
\alpha_{ij} = \frac{1}{2} \left[ 1 - \sum_{k \in N_i \setminus \{j\}} \alpha_{ik} + \frac{\rho}{1+\rho} \left( \sum_{k \in N_i \setminus \{j\}} \alpha_{ki} - \sum_{k \in N_i \setminus \{i\}} \alpha_{kj} \right) \right] \forall ij \text{ s.t. } G_{ij} = 1 \quad (14)
\]

In equation (14), the net transferred shared \( \alpha_{ij} \) of \( e_i \) from \( i \) to \( j \) is given by the half of the “remaining share” after deducting the transfers to \( i \)’s other friends \( N_i \setminus \{j\} \), corrected by an adjustment for inflows of non-local shocks. The \( \frac{1}{2} \) multiplier is analogous to the equal sharing rule in the independent endowments case, but last term in the square brackets is new.\(^{12}\) We refer to it as an informational effect, for the following reason. \( \sum_{k \in N_i \setminus \{j\}} \alpha_{ki} \) is the sum of \( i \)’s shares of \( i \)’s other neighbors’ endowments \( (e_k)_{k \in N_i \setminus \{j\}} \), and the conditional expectation of each \( k \)’s endowment changes linearly with the realization of \( e_i \) by a factor of \( \frac{\rho}{1+\rho} \). Similarly, \( \sum_{k \in N_i \setminus \{i\}} \alpha_{kj} \) is the sum of \( j \)’s shares of \( j \)’s other neighbors’ endowments \( (e_k)_{k \in N_j \setminus \{i\}} \), and the conditional expectation of each \( k \)’s endowment also changes linearly with the realization of \( e_i \) by a factor of \( \frac{\rho}{1+\rho} \). Due to the symmetric correlation structure, the realization of \( e_i \) provides the same amount of local information about all non-local endowment realizations \( e_k \) for \( k \not\in N_i \cap N_j \), and thus its informational effect can be calculated as a simple net summation of endowment shares. As a higher realized \( e_i \) predicts that both \( i \) and \( j \) are more likely to obtain higher amounts of inflows\(^{13}\) from uncommon neighbors, this commonly recognized information can be used by the pair \( ij \) (imperfectly) share the non-local risk exposures. After pooling the conditional expectations of non-local inflows, \( i \) and \( j \) again share the remaining shares of \( e_i \) and \( e_j \) equally. It is worth pointing out that \( i \) carries out this kind of “equal sharing” with all her neighbors, and the inflow/outflow shares \( \{\alpha_{ij}\} \) must make all the sharing simultaneously equal (in expectation).

Hence, the \( (\sum_{i \in N} d_i) \)-dimensional vector \((\alpha_{ij})_{G_{ij}=1}\) must solve the system of \( (\sum_{i \in N} d_i) \) linear equations defined by (14). No more linear restrictions need to be imposed on \((\alpha_{ij})_{G_{ij}=1}\), because, for each \( i \), \( i \) herself absorbs \( 1 - \sum_{j \in N_i} \alpha_{ij} \) of her own endowment shock so that \( e_i \) is fully shared within \( i \)’s extended neighborhood. Writing each equation in (14) in the canonical form

\[
2\alpha_{ij} + \sum_{k \in N_i \setminus \{j\}} \alpha_{ik} - \sum_{k \in N_i \setminus \{j\}} \frac{\rho}{1+\rho} \alpha_{ki} + \sum_{h \in N_j \setminus \{i\}} \frac{\rho}{1+\rho} \alpha_{hi} = 1 \quad \forall ij \text{ s.t. } G_{ij} = 1
\]

\[
\Leftrightarrow \qquad A \alpha = 1_{\sum_i d_i}
\]

It can be shown that this system has a solution, but we leave this to be established later, for general networks.

\(^{12}\)This term disappears when \( \rho = 0 \).

\(^{13}\)To be precise, by “inflow” we mean the undertaking of a share of someone else’s income endowment, which may be positive or negative; by “outflow” we mean the distribution of a share of one’s own endowment to someone else, which may also be positive or negative. In particular, a negative inflow is not the same as an outflow. Instead, \( i \)’s inflow from \( j \) is the same as \( j \)’s outflow to \( i \).
For general network structure, the analysis is similar to the above, but there are several complications. As $I_{ij} = (e_i, e_j, e_{N_{ij}})$, the transfer rule $t_{ij}$ can be contingent on $e_{N_{ij}} := (e_k)_{k \in N_{ij}}$ in addition to $e_i, e_j$. Furthermore, as the knowledge of the ex post realization of $e_{N_{ij}}$ brings in extra information about the distribution of non-local endowment realizations, Pareto efficiency requires that $t_{ij}$ be contingent on $e_{N_{ij}}$. Specifically,

$$e_k|_{e_i, e_j, e_{N_{ij}}} \sim N \left( \frac{\rho}{1 + (d_{ij} + 1) \rho} \left( e_i + e_j + \sum_{k \in N_{ij}} e_k \right), V_{d_{ij} + 2} \right)$$

where $d_{ij} := \#(N_{ij})$ and $V_{d_{ij} + 2}$ denotes the variance of $e_k$ conditional on observing $(d_{ij} + 2)$ endowment realizations.

Again, we postulate a linear transfer rule:

$$t_{ij} = \alpha_{ij} e_i - \alpha_{ji} e_j + \sum_{k \in N_{ij}} \beta_{ijk} e_k + \mu_{ij} \quad \forall G_{ij}=1.$$  

Then, after some tedious algebraic transformations\footnote{See Lemma 7 in the Appendix.}, we arrive at the following system of FOCs for Pareto efficiency: \( \forall ij \) s.t. $G_{ij} = 1$,

\[
\begin{align*}
\alpha_{ij} &= \frac{1}{2} \left( 1 - \sum_{k \in N_i \setminus \{j\}} \alpha_{ik} + \sum_{k \in N_{ij}} \beta_{jki} + \gamma_{ij} \right) \\
\beta_{ijk} &= \frac{1}{2} \left( \alpha_{ki} - \alpha_{kj} + \sum_{h \in N_{ijk}} (\beta_{ikh} - \beta_{jkh}) \\
&\quad - \sum_{h \in N_{ikj} \setminus N_j} \beta_{ikh} + \sum_{h \in N_{njk} \setminus N_i} \beta_{jkh} + \gamma_{ij} \right) \quad \forall k \in N_{ij} \\
\gamma_{ij} &= \frac{\rho}{1 + (d_{ij} + 1) \rho} \left[ \sum_{k \in N_{ik} \setminus N_j} \left( \alpha_{ki} - \sum_{h \in N_{ikj} \setminus N_j} \beta_{ikh} \right) \\
&\quad - \sum_{k \in N_{ij} \setminus N_i} \left( \alpha_{kj} - \sum_{h \in N_{njk} \setminus N_i} \beta_{jkh} \right) \\
&\quad - \sum_{k \in N_{ij} \setminus N_{ikj} \setminus N_i} \left( \sum_{h \in N_{ikj} \setminus N_j} \beta_{ikh} - \sum_{h \in N_{njk} \setminus N_i} \beta_{jkh} \right) \right] \\
\end{align*}
\]

We interpret $\gamma_{ij}$ as the net informational effect because it is the rate at which locally observed endowment realization affects the pair $ij$’s joint expectation of non-local endowments. As each element of $(e_k)_{k \in N_{ij}}$ provides exactly the same amount of information to the linked pair $ij$ for their joint inference on non-local endowments, $\gamma_{ij}$ is the same across $k \in N_{ij}$.

The system \(15\) consists of \(2 \sum_i d_i + \sum_{G_{ij}=1} d_{ij}\) linear equations. Due to the possible existence of cycles, multiple solutions may arise by adding superfluous transfers along cycles. For example, given a complete triad $ijk$, we can make a superfluous transfer of a $\epsilon$ share of $e_i$ from $i$ to $j$, $j$ to $k$ and $k$ to $i$ by adding $\epsilon$ to $\alpha_{ij}, \beta_{jki}$, and subtracting $\epsilon$ from $\alpha_{ik}$. It can then be checked that this operation is indeed superfluous, in the sense that $(\alpha_{ij} + \epsilon, \beta_{jki} + \epsilon, \beta_{kji} - \epsilon, \alpha_{ik} - \epsilon)$ along with all other elements of $\alpha, \beta, \gamma$ kept unchanged still solve the system \(15\), with the final consumption allocations left exactly the same as before. With this degree of freedom to implement any amount of superfluous transfers along all cycles, we can set $\beta_{ijk} = 0$ for all triads $ijk$ without loss of Pareto efficiency. In the
following, we establish that there exists some $\alpha$ such that $(\alpha, \beta = 0)$ solves $15$ and thus achieves Pareto efficiency.

By setting $\beta = 0$, we can achieve significant simplification of $15$ and obtain the following system:

$$
\begin{align*}
0 &= \alpha_{ki} - \alpha_{kj} + \gamma_{ij} \quad \forall k \in N_{ij} \\
\gamma_{ij} &= \frac{1}{1 + (d_{ij} + 1)\rho} \left( \sum_{k \in N_i \setminus N_j} \alpha_{ki} - \sum_{k \in N_j \setminus N_i} \alpha_{kj} \right)
\end{align*}
$$

The first equation $16.1$ states that the share of $e_i$ transferred from $i$ to $j$ is half of the remaining share after $i$’s transfers to $i$’s other friends plus the informational adjustment term between $ij$. With $\gamma \equiv 0$, which is implied by $\rho = 0$, $\alpha$ will be simply reduced to the local equal sharing rule. The second equation $16.2$ requires that the difference in the shares of $e_k$ undertaken by $i$ and $j$ is equal to the informational effect between $ij$, so that it is indeed optimal for $ij$ to set $\beta_{ijk} = 0$. Hence, strict bilaterality $(\beta = 0)$ is not an assumption, because $16.2$ incorporates the efficiency requirements for $\beta = 0$. The third equation $16.3$ defines the informational effect between $ij$ as the differences between the sums of uncommon inflow shares to $i$ and $j$. Given $\alpha$, $|\gamma_{ij}|$ is decreasing in $d_{ij}$, indicating that the magnitude of the informational effect is decreasing in the amount of local information. Below we proceed to show the existence of a solution to $16$.

Plugging $16.3$ into the $16.1$, we obtain a system of equations in $\alpha$ only:

$$
\alpha_{ij} = \frac{1}{2} \left( 1 - \sum_{k \in N_i \setminus \{j\}} \alpha_{ik} + \frac{\rho}{1 + (d_{ij} + 1)\rho} \left( \sum_{k \in N_i \setminus N_j} \alpha_{ki} - \sum_{k \in N_j \setminus N_i} \alpha_{kj} \right) \right) \quad \forall i, j \text{ s.t. } G_{ij} = 1
$$

Equation $17$ looks very similar to $14$ with changes reflecting the generality of the network relative to tree networks: $d_{ij}$ reflects the number of additional sample points for joint inference about non-local endowment realizations, while $N_i \setminus N_j$ and $N_i \setminus N_i$ are general representations of the sets of uncommon neighbors. Writing $17$ in the canonical form, we get

$$
2\alpha_{ij} + \sum_{h \in N_i \setminus \{j\}} \alpha_{ih} - \frac{\rho}{1 + (d_{ij} + 1)\rho} \sum_{h \in N_i \setminus N_j} \alpha_{hi} + \frac{\rho}{1 + (d_{ij} + 1)\rho} \sum_{h \in N_j \setminus N_i} \alpha_{hj} = 1
$$

and equivalently in matrix notations

$$
A\alpha = 1.
$$

At $\rho = 0$, $\det (A) > 0$ and there always exist a solution to the system, given by the local equal sharing rule $\alpha_{ij} = \frac{1}{d_{ij} + 1}$. Notice that

$$
\det (A(\rho)) = \prod_{G_{ij} = 1} \frac{P(\rho)}{[1 + (d_{ij} + 1)\rho]^{r_{ij}}}
$$
where $P(\rho)$ is a finite-order polynomial in $\rho$. For $\rho \in \Lambda := \left[ -\frac{1}{n-1}, 1 \right]$, the denominator is always strictly positive and bounded away from 0, so that $\det(A)$ is always well-defined in $\mathbb{R}$, and it is also continuous in $\rho$ and bounded on $\left[ -\frac{1}{n-1}, 1 \right]$. Define

$$\Lambda_0 := \{ \rho \in \Lambda : \det(A(\rho)) = 0 \}.$$

By the Fundamental Theorem of Algebra, $\#(\Lambda_0) < \infty$. For $\rho \in \Lambda \setminus \Lambda_0$, the solution is well-defined as $\alpha(\rho) := A(\rho)^{-1} 1$, and $\alpha(\rho)$ is continuous on $\Lambda \setminus \Lambda_0$.

However, for $\alpha(\rho)$ to be Pareto efficient, (16.2) must also be satisfied. By differencing (16.1) for $k_i$ and for $k_j$ we get: $\alpha_{ki} - \alpha_{kj} = \gamma_{ki} - \gamma_{kj}$. Hence, in the presence of (16.1) equation (16.2) is equivalent to, for all triads $ijk$,

$$\gamma_{ij} + \gamma_{jk} + \gamma_{ki} = 0. \quad (18)$$

This is reminiscent of the Kirchhoff Voltage Law for electric resistor networks, which states that the sum of voltage differences across any closed cycle must sum to zero. It turns out that the Kirchhoff Voltage Law indeed holds in our setting, so that the equality (18) can be generalized to any cycle in a general network.

**Proposition 4. The “Kirchhoff Voltage Law”:** $\forall \rho \in \Lambda \setminus \Lambda_0$, (16.1) and (16.3) imply (16.2). Furthermore, given any cycle $i_1i_2...i_mi_1$, we have

$$\gamma_{i_1i_2} + \gamma_{i_2i_3} + ... + \gamma_{i_mi_1} = 0.$$

**Proof.** See the Appendix.

Given the redundancy of (16.2) in the presence of (16.1) and (16.3), as established in Proposition 4, we may now conclude that, $\forall \rho \in \Lambda \setminus \Lambda_0$, $\alpha(\rho)$ satisfies all optimality conditions in system (16) and is thus Pareto efficient. For $\rho \in \Lambda_0 \setminus \{1\}$, we may take a sequence $\{\rho_m\} \subseteq \Lambda \setminus \Lambda_0$ that converges to $\rho$ and show that the limit of (a subsequence of) $\alpha(\rho_m)$ also solves system (16) and is thus also Pareto efficient. For $\rho = 1$, $\alpha \equiv 0$ is trivially Pareto efficient; moreover, any profile of $\alpha$ that satisfies the “Kirchhoff Circuit Law”, i.e., $\sum_{j \in N_i} (\alpha_{ij} - \alpha_{ji}) = 0 \forall i \in N$, is Pareto efficient. The following proposition summarizes the findings.

**Proposition 5.** $\forall \rho \in \left[ -\frac{1}{n-1}, 1 \right]$, there exists a linear, strictly bilateral, and Pareto efficient profile of transfer rules.

**Proof.** See the Appendix.

In general, the explicit characterization of Pareto efficient profile of transfers by $\alpha = A(\rho)^{-1} 1$ involves taking the inverse of matrix $A(\rho)$, which is complicated for general networks.\(^\text{15}\) However, for symmetric networks, that is when the matrix representation of the

\(^{15}\)See the next subsection for an explicit characterization for star networks.
network is permutationally symmetric, regardless of the correlation \( \rho \), we get back the local equal sharing rule:

\[
t_{ij}^* (e_i, e_j) := \frac{e_i}{d+1} - \frac{e_j}{d+1} + \mu_{ij},
\]

with \( d \) being the common degree for each agent. Potential differences on a symmetric network become \( \gamma_{ij} = 0 \) along each link \( ij \), because the nonlocal net exposures for \( i \) and \( j \) will exactly cancel each other out. Simultaneously, given that \( \gamma = 0 \), the local equal sharing rule \( t^* \) constitutes the unique solution for system \((16)\).

## 4.3 Structural Properties of Efficient Risk Sharing with Local Information

Below we analyze the properties of the Pareto efficient transfer arrangements characterized in the previous subsection in more detail.

In observation of Proposition 4, \( \forall \rho \neq 0 \), we can define a potential function \( V : N \rightarrow \mathbb{R} \) s.t.

\[
V(i) - V(j) = \frac{1}{\rho} \cdot \gamma_{ij} = \frac{1}{1 + (d_{ij} + 1) \rho} \left( \sum_{k \in N_i \setminus N_j} \alpha_{ki} - \sum_{k \in N_j \setminus N_i} \alpha_{kj} \right).
\]

In particular, \( V \) can be constructed from \( \gamma \) in the following way. Fix any individual, say, individual 1, and normalize \( V(1) = 0 \). For any other individual \( j \neq 1 \), as the network is connected, there exists a path \( 1i_1...i_mj \) that connects 1 to \( j \). Then we simply define \( V(j) = \frac{1}{\rho} \cdot (\gamma_{1i_1} + ... + \gamma_{1m_j}) \). By Proposition 4, \( V \) is well-defined.

Using terminology from resistor networks, \( V(i) \) is analogous to an “energy potential” for node \( i \), and \( \gamma_{ij} \) is the “potential difference” or “voltage” between \( i \) and \( j \). Given \( \gamma \), the “current flows” \( \alpha \) (or more precisely, its deviation from local equal sharing) are driven by \( \gamma \) according to \((16.1)\). However, \( \gamma \) is simultaneously determined by the currents \( \alpha \) according to \((16.3)\). The multiplier \( \frac{\rho}{1+(d_{ij}+2)\rho} \) in the \( \Gamma \) matrix captures how much a pair \( ij \) “discounts” nonlocal inflows given the amount \( (d_{ij} + 2) \) and the quality \( (\rho) \) of local information. Despite the possible differences in this discount multiplier across linked pairs, the optimum is associated with a globally consistent assessment of each individual’s “net position in nonlocal risk exposures”, and it is summarized by the “potential centrality” \( V(i) \).

Denote

\[
\alpha_{ii} := 1 - \sum_{k \in N_i} \alpha_{ik},
\]

which is simply agent \( i \)'s net exposure to her own income shock. Then from Proposition 4 we can immediately derive the following corollary.

**Corollary 2.** Take any \( \rho \in \Lambda \setminus \Lambda_0 \) and suppose that the system \((16)\) holds. Then, for any linked \( ij \) with \( V(i) - V(j) > 0 \), we have

\[
\alpha_{ki} - \alpha_{kj} < 0
\]

for any \( k \in \{i, j\} \cup N_{ij} \).
In words, if \(i\) has a higher “potential” than \(j\), then \(i\) gets a larger net exposure to all the commonly observable endowment shocks than \(j\). This is achieved by a larger transfer of \(e_i\) from \(i\) to \(j\) and a smaller transfer of \(e_j\) from \(j\) to \(i\) relative to local equal sharing. Moreover, the net exposure difference to any particular commonly observable shock will be exactly the same and given by \(\gamma_{ij}\). As a result, \(i\) and \(j\)’s difference in net exposures from commonly observable shocks will offset their expected difference in their uncommon shocks.

In economic terms, \(V(i)\) can be interpreted as the effective centrality of agents in risk-sharing activities, given risk sharing is Pareto efficient subject to local information constraints: a higher \(V(i)\) is indicative of more inflow exposures relative to \(i\)’s neighbors’ inflows, as well as a relative scarcity of “alternative paths” around \(i\) (recall that \(|\gamma_{ij}|\) decreases in \(d_{ij}\)).

The fact that in a Pareto optimal arrangement neighboring agents equally share expected state-dependent shocks conditional on their common information does not mean though that their ex ante consumption variance is equal. The transfer scheme that achieves the equalization of conditional expectations of the state-dependent part of the final consumptions has the feature that the neighbor with a higher exposure to non-common shocks ends up with higher consumption variance. Mathematically this is because conditional expectations of uncommon shocks are equal to a constant \(\frac{\rho}{1+(d_{ij}+1)\rho}\) times the sum of common shocks, and this constant is strictly smaller than 1. Hence, a unit increase in exposure to non-common shocks is compensated by less than a unit decrease in exposure to common shocks. This implies that agents who are more central in risk sharing end up with a higher consumption variance in our model.

We illustrate the above relationships between network centrality, transfer flows and consumption variances in the context of star networks. Let \(c\) denote the center agent, who is connected to \(n-1\) peripheral agents, and none of the peripheral agents are connected to each other. We use \(p\) to refer to a generic peripheral agent.

It is straightforward to show that a linear transfer arrangement achieving Pareto efficiency subject to local information constraints specifies the following endowment shares to be transferred:

\[
\alpha_{cp} = \frac{2 + 2(n - 1)\rho}{n(2 + n\rho)}, \quad \alpha_{pc} = \frac{1 + \rho}{2 + n\rho},
\gamma_{cp} = \frac{(n - 2)\rho}{2 + n\rho}.
\]

Given this transfer arrangement, if the potential centrality of peripheral agents is normalized to 0, the potential centrality of the center is \(V(c) = \frac{n-2}{2+n\rho}\). Final consumptions are given by:

\[
x_c = \frac{2(1+\rho) - (n-2)^2\rho}{n(2 + n\rho)}e_c + \frac{1 + \rho}{2 + n\rho}\sum_{k \neq c} e_k + C_i
\]

\[
x_p = \frac{2 + 2(n - 1)\rho}{n(2 + n\rho)}e_c + \frac{1 + (n - 1)\rho}{2 + n\rho}e_p + C_p.
\]

Hence the difference in consumption variances satisfies

\[
Var(x_c) - Var(x_p) = \frac{(n - 2) (1 + (n - 1) \rho) (1 - \rho^2)}{(2 + n\rho)^2} \geq 0
\]
with equality only at $\rho \in \{-\frac{1}{n-1}, 1\}$. In particular, $Var(x_c) - Var(x_p) \rightarrow \frac{1-\rho^2}{\rho}$ as $n \rightarrow \infty$, and hence the consumption variance of the center can be much higher than the consumption variance of a periphery agent when $\rho$ is low and $n$ is high.

Centrality in risk sharing, as implied via the Pareto efficient transfer arrangements subject to local information constraints, is not equivalent to standard notions of centrality, such as degree or eigenvector centrality. However, on typical networks they are highly positively correlated, implying that our model predicts a positive relationship between these centrality measures and consumption variance. This contrasts with the predictions of the model in AMS, in which enforcement constraints limit the efficiency of risk-sharing arrangements. We illustrate this point numerically, estimating the correlation predicted by our model and by the model in AMS, between an agent’s centrality and consumption variance, via simulated shocks in two real-world village networks from India from two different databases, each randomly selected and provided us by the researchers who collected the data.

In both simulations, we randomly drew the endowment $e_{i(t)}$ of each household according to the standard normal distribution for $T = 5000$ times: $\{e_{i(t)}\}_{i,t} \sim iid \ N(0, 1)$. We assumed that all households have CARA utility functions with $\lambda = 1$. We then computed the final consumptions of each household under the equally-weighted Utilitarian optimal transfer arrangement subject to local information constraints, using the results from subsection 4.1, and the sample variance of final consumptions for each household (note that the variance does not depend on the planner’s weights). Following this we computed the sample correlation between degree/eigenvector centrality and consumption variance. Similarly, we computed the constrained efficient consumptions implied by the model in AMS, and the sample correlation between the centrality measures and consumption variance under three levels of capacity constraints (the maximum amount that can be transferred through any link, at any state): 0.5, 1 and 1.5. The results are summarized in the following table. All results are highly statistically significant with p-values very close to zero, except for the correlation between consumption variance and eigenvalue centrality in the model of our paper when using the BCDJ data.

---

16 The first network was provided to us by Erica Field and Rohini Pande, who collected it from villages in the districts of Thanjavur, Thiruvarur and Pudukkotai (Tamil Nadu) in India. In a subset of the villages, complete within-village network data was collected by surveying all households. The second network is from data collected by Banerjee, Chandrasekhar, Duflo and Jackson in Karnataka, India (they collected complete within-village network data in 75 villages), used for example in the Banerjee, Chandrasekhar, Duflo, and Jackson (2014). From both datasets we received the network of financial connection for one randomly selected village with complete network data. From the original network we created the undirected “AND” network, that is, we defined a link between two households whenever both households indicated each other as a borrowing relationship. We excluded households that became isolated in the “AND” network.

17 The p-values, calculated from standard t-tests against the null hypotheses of zero correlations, are at orders of magnitudes below $10^{-10}$, except the case noted.
Table 1: Correlation between Centralities and Consumption Variances

<table>
<thead>
<tr>
<th></th>
<th>(A) Field &amp; Pande</th>
<th></th>
<th>(B) BCDJ</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Capacity</td>
<td>Degree</td>
<td>Eigenvector</td>
</tr>
<tr>
<td>AMS</td>
<td>0.5</td>
<td>-0.8943***</td>
<td>-0.4585***</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>-0.6885***</td>
<td>-0.2313***</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>-0.5430***</td>
<td>-0.0898***</td>
</tr>
<tr>
<td>AG</td>
<td></td>
<td>0.1994***</td>
<td>0.1659***</td>
</tr>
</tbody>
</table>

*** denotes statistical significance at 1%-level.

Under the AMS model, we observe a negative correlation between centrality and consumption variance. In AMS, transfers along links are subject to capacity constraints. As a result, centrally located households tend to have a lower consumption variance, because capacity constraints are less likely to be binding for them locally, and for typical shock realizations they end up pooling risk with a larger set of other households.

This holds for all capacity values we used in the simulations, but the relationship is more highlighted for relatively stricter capacity constraints.

Under the model of the current paper, we observe the opposite sign: sample correlation between both degree and eigenvector centrality on the one hand, and consumption variance on the other hand is positive (as noted above, not significantly for eigenvalue centrality when using the BCDJ data).

5 Discussion

5.1 The effect of the spatial structure of correlation

In the previous section we considered a symmetric correlation structure, in which the correlation between the endowment realizations of two agents did not depend on their positions on the network. An alternative specification, however, is to incorporate the possibility of spatially correlated shocks, that is correlation that decays with social distance. As we demonstrate below, this type of correlation structure can be detrimental to the efficiency of informal risk sharing with local information constraints.

Specifically, we assume that the correlation between $e_i$ and $e_j$ geometrically decays with social distance between $i$ and $j$:

$$\text{Corr} (e_i, e_j) = g^{\text{dist}(i,j)},$$

18 Using terminology from AMS, more centrally located households typically end up on larger “risk-sharing islands.”

19 As capacities increase, centrality in the AMS model matters less, since capacity constraints are less likely to bind.

20 There are many reasons why this correlation structure is more realistic for certain types of shocks: for example, as shown in Fafchamps and Gubert (2007b) and in Conley and Udry (2010), social distance tends to be highly correlated with geographic proximity.
where the social distance \( \text{dist}(i, j) \) is formally defined as the length (i.e., the number of links) of the shortest path connecting \( i \) and \( j \) in network \( G \). For tractability, we restrict attention to circle networks.

A \( n \)-circle consists of \( n \) individuals and \( n \) links: \( G_{i,i+1} = 1 \) for \( i = 1, \ldots, n \). For any linked pair \( i, i + 1 \) along a \( n \)-circle (with \( n \geq 4 \)), the conditional distribution of \( e_{i-1} \) (and similarly for \( e_{i+2} \)) is

\[
e_{i-1} | e_i, e_{i+1} \sim N \left( \varrho e_i, (1 - \varrho) \sigma^2 \right).
\]

Following a similar argument as in Section 4.2, we obtain the following condition for Pareto efficiency subject to local information constraints:

\[
\begin{align*}
\alpha_{i,i+1} &= \frac{1}{2} \left( 1 - \alpha_{i-1,i} + \varrho \alpha_{i-1,i} \right) \\
\alpha_{i+1,i} &= \frac{1}{2} \left( 1 - \alpha_{i+1,i+2} + \varrho \alpha_{i+1,i+2} \right)
\end{align*}
\]

for all \( i \in N \). Then, the unique and symmetric solution for the above system is given by

\[
\alpha_{ij}^* \equiv \alpha_{geo}^* (\varrho) = \frac{1}{3 - \varrho} \quad \forall G_{ij} = 1.
\]

Under \( \alpha^* \), the final consumption for each individual is

\[
x_{i}^{geo} (\varrho) = \frac{1}{3 - \varrho} e_{i-1} + \frac{1}{3 - \varrho} e_i + \frac{1}{3 - \varrho} e_{i+1}
\]

with a variance of

\[
\text{Var}_{geo,\varrho} (x_{i}^{geo} (\varrho)) = \frac{1 + \varrho}{3 - \varrho}.
\]

In comparison, under the symmetric correlation structure in Section 4.2, the condition for Pareto efficiency on a \( n \)-circle is

\[
\alpha_{i,i+1} = \frac{1}{2} \left[ 1 - \alpha_{i-1,i} + \frac{\varrho}{1 + \varrho} (\alpha_{i-1,i} - \alpha_{i+2,i+1}) \right]
\]

with its unique and symmetric solution being

\[
\alpha_{ij} \equiv \alpha_{unif} (\varrho) = \frac{1}{3} \quad \forall G_{ij} = 1,
\]

which is exactly the local equal sharing rule. This implies a final consumption of

\[
x_{i}^{unif} (\varrho) = \frac{1}{3} e_{i-1} + \frac{1}{3} e_i + \frac{1}{3} e_{i+1}
\]

with a variance of

\[
\text{Var}_{unif,\varrho} (x_{i}^{unif} (\varrho)) = \frac{1 + 2\varrho}{3}.
\]

\footnote{We, for notational simplicity, define agent \( n + 1 \) to be agent 1, and agent 0 to be agent \( n \).}
We compare the correlation structures by setting \( \rho \) and \( \varrho \) to be such that each individual’s final consumption variance is equalized across the two correlation structures under the global equal sharing rule (which achieves first best risk sharing):

\[
x_i^{FB} = \frac{1}{n} \sum_{k \in N} e_k.
\]

The consumption variances that this sharing rule implies for the two correlation structures are:

\[
Var_{\text{unif}, \rho} (x_i^{FB}) = \frac{1 + (n - 1) \rho}{n},
\]

\[
Var_{\text{geo}, \varrho} (x_i^{FB}) = \begin{cases} 
\frac{1 + 2 \sum_{k=1}^{m-1} \varrho^k + \varrho^m}{n}, & n = 2m \\
\frac{1 + 2 \sum_{k=1}^{m} \varrho^k}{n}, & n = 2m + 1 \\
\frac{1 - \varrho^{m+1} + \varrho^m}{1 - \varrho}, & n = 2m \\
\frac{1 - \varrho^{m+1} + \varrho^m}{1 - \varrho}, & n = 2m + 1
\end{cases}
\]

For simplicity focus on \( n = 2m + 1 \). Then

\[
Var_{\text{unif}, \rho} (x_i^{FB}) = Var_{\text{geo}, \varrho} (x_i^{FB}) \iff \frac{1 + (n - 1) \rho}{n} = \frac{2 \varrho^{m+1} - 1}{n} \iff \rho = \rho (\varrho) := \frac{\varrho (1 - \varrho^m)}{m (1 - \varrho)}.
\]

In short \( \rho = \rho (\varrho) \) implies that the total amount of sharable risk is equalized between the two correlation structures. Next we compare the consumption variances given Pareto efficient risk-sharing arrangements subject to local information constraints.

Notice that

\[
Var_{\text{unif}, \rho} (x_i^{\text{uni}f} (\rho)) \leq Var_{\text{geo}, \varrho} (x_i^{\text{geo}^f} (\varrho)) \iff \rho \leq \overline{\varrho} (\varrho) := \frac{2 \varrho}{3 - \varrho}.
\]

Hence, whenever

\[
m > \frac{(3 - \varrho) (1 - \varrho^m)}{2 (1 - \varrho)}
\]

we will have \( \rho (\varrho) < \overline{\varrho} (\varrho) \) and thus \( Var_{\text{uni}f}^\rho (x_i^{\text{uni}f} (\rho)) < Var_{\text{geo}}^\varrho (x_i^{\text{geo}^f} (\varrho)) \). In other words, fixing \( \varrho \), efficient risk sharing subject to the local information constraint performs strictly better under the uniform correlation setting than under the geometrically decaying setting.

Moreover, the difference can be very stark. As \( m \to \infty \),

\[
\rho = \rho (\varrho) = \frac{\varrho (1 - \varrho^m)}{m (1 - \varrho)} \to 0
\]

and thus

\[
Var_{\text{unif}, \rho} (x_i^{\text{uni}f} (\rho)) = \frac{1 + 2 \rho}{3} \to 0,
\]

26
while
\[ \text{Var}_{\text{geo}, \rho} (x_{i}^{\text{geo}} (\varrho)) = \frac{1 + \varrho}{3 - \varrho} \quad \forall m. \]

When also taking \( \varrho \to 1 \) (after taking \( m \to \infty \)), we get
\[
\lim_{\varrho \to 1} \lim_{m \to \infty} \text{Var}_{\text{unif}, \rho (\varrho)} (x_{i}^{\text{unif}} (\rho (\varrho))) = 0,
\]
\[
\lim_{\varrho \to 1} \lim_{m \to \infty} \text{Var}_{\text{geo}, \rho} (x_{i}^{\text{geo}} (\varrho)) = 1.
\]

Hence, for \( \varrho \) close to 1, for sufficiently \( m \), uniform correlation leads to almost perfect risk sharing, while geometrically decaying correlation yields payoffs very close to the autarky payoffs, even though the two correlation structures lead to the same payoffs if global information can be used for risk sharing. This might help explain why it is the case that while in most settings empirical research found that informal insurance works well, Kazianga and Udry (2006) found a setting in which informal insurance does not seem to help, and Goldstein, de Janvry, and Sadoulet (2001) found that certain types of shocks are not well insured through informal risk sharing. In particular, this may be due to high correlation between shocks of neighboring households in the above settings, for the types of shocks investigated.

5.2 Endogenous Network Formation

So far our analysis focused on characterizing Pareto efficient risk-sharing arrangements subject to local information constraints on an exogenously given network, implicitly assuming that network connections are mainly shaped by predetermined factors such as kinship. Here we briefly discuss some implications of allowing for endogenous link formation in the context of informal risk sharing with local information constraints. The approach we take is similar as in Ambrus, Chandrasekhar, and Elliott (2015), who consider network formation in a risk-sharing framework with global information contracts, and propose a two-stage game in which in the first stage agents can simultaneously indicate other agents they want to link with. If two agents each indicated each other, the link is formed, and the two connecting agents each incur a cost of \( c \geq 0 \).\(^{22}\) In the second stage, whatever network is formed in the first stage, it is assumed that agents agree on a Pareto efficient risk-sharing arrangement subject to local information constraints (more on this below).

In the first stage game the solution concept we use is pairwise stability, implying that for any established link neither of the two agents has a strict incentive to unilaterally drop it (not establishing it), and that for any potential link not established, it cannot be that both of the agents would prefer to establish the link, with at least one of them strictly.

In our analysis of the CARA-normal framework so far, state independent transfers played a very limited role. However, when we allow for endogenous network formation, it becomes crucial how the network structure influences state independent transfers, and hence the distribution of surplus created by risk sharing, as it directly affects incentives to form links.

\(^{22}\)This simple game of network formation was originally considered in Myerson (1991). See also Jackson and Wolinsky (1996).
Therefore, it is important to specify exactly which Pareto efficient risk-sharing arrangement prevails for each possible network that can form. Different ways of specifying state-independent transfers can lead to very different conclusions regarding network formation, as we demonstrate below.

A benchmark case is when all state-independent transfers are set to 0, motivated by the consideration that in social networks people might not tend to pay or collect “risk premium” when engaging in informal risk-sharing interactions. This case is extensively investigated by Gao and Moon (2016) who assume local equal sharing with no state-independent transfers as an ad hoc sharing rule. They show that, even without cost of linking, an agent \(i\)’s benefit for establishing an extra link with \(j\) falls very fast with the existing number of links the agent \(i\) has, with more existing neighbors (larger \(d_i\)) the marginal reduction in self-shock exposure \(\left(\frac{1}{d_i+1} - \frac{1}{d_i+2}\right)\) is small relative to the additional exposure to \(j\)’s shock \(\frac{1}{d_j+2}\). In fact, this diminishing return from linking is so severe that stable networks are almost 2-regular, with an upper bound of 2 on the average degree. Typically this implies severe underinvestment into social connections, in the sense that there are many links that are not established in equilibrium, even though they would be socially beneficial.

Another benchmark case to consider is the equal-weight Utilitarian transfer. Under this setting, the Pareto efficient state-independent transfers will equally redistribute the ex ante surplus from risk sharing to each individual in the network, leading to egalitarian expected payoffs. Hence, without the cost of linking, the unique Pareto efficient network will be the efficient complete graph. However, when the cost of linking becomes large, the usual hold-up problem emerges: the pair who pay the cost of linking cannot capture the whole social surplus, leading to underinvestment inefficiencies.

An alternative approach is pursued by Ambrus, Chandrasekhar, and Elliott (2015), in the context of risk-sharing arrangements in a CARA-normal: they assume that the profile of state-independent transfers is determined according to the Myerson value. The Myerson value, proposed in Myerson (1980), is a network-specific version of the Shapley value that allocates surplus according to average incremental contribution of agents to total social surplus. Ambrus, Chandrasekhar, and Elliott (2015) in particular show that with state-independent transfers specified as above (for whatever network is formed), if agents are ex ante symmetric then there is never underinvestment, that is given any stable network, there is no potential link that is not established, even though its net social value would be strictly positive. Below we show that the same conclusion holds in our setting with local information constraints, in the case of CARA utilities and independently an jointly normally distributed endowments.

In our setting, for a given network \(G\), individual \(i\)’s Myerson value is defined by

\[
MV_i(G) := \sum_{S \subseteq N} \frac{(\#(S) - 1)(n - \#(S))}{n!} \cdot \frac{1}{2} \sigma^2 \left[ TVar \left( G \mid (S \setminus \{i\}) \right) \right] + \sigma^2 - TVar \left( G \mid S \right)
\]

where \(\#(S)\) denotes the number of individuals in a subset \(S\) of \(N\), and \(G \mid S\) denotes the subgraph of \(G\) restricted to the subset \(S\) of individuals. Given the CARA-normal setting, Ambrus, Chandrasekhar, and Elliott (2015) also provide micro-foundations, in the form of a decentralized bargaining procedure between neighboring agents that leads to state independent transfers achieving the Myerson value allocation.
TVaR \( G|_{S \setminus \{i\}} \) + \sigma^2 - TVaR \( G|_S \) is the surplus from risk reduction through \( i \)'s links in \( S \).

Notice that, given any \( S \subseteq N \),

\[
TVaR \left( G|_{S \setminus \{i\}} \right) - TVaR \left( G|_S \right) = 1 - \frac{1}{d_i(G|_S) + 1} + \sum_{k \in N_i(G|_S)} \frac{1}{d_k(G|_S) [d_k(G|_S) + 1]},
\]

which is strictly increasing in \( d_i(G|_S) \) but strictly decreasing in \( d_k(G|_S) \) for each \( j \in N_k(G|_S) \). Moreover, for any \( k \in N, d_k(G|_S) \) is weakly increasing in \( S \), i.e., \( S \subseteq S' \Rightarrow d_k(G|_S) \leq d_k(G|_{S'}) \).

Consider any pairwise stable network \( G \) under the Myerson-value transfers. Then, if \( i, j \) are linked, it must be that

\[
MV_i(G) - MV_i(G - ij) \geq c.
\]

Fixing \( ij \), for each \( S \subseteq N \), we have

\[
\begin{align*}
TVaR \left( G - ij|_{S \setminus \{i\}} \right) - TVaR \left( G - ij|_S \right) &= \begin{cases} 
TVaR \left( G|_{S \setminus \{i\}} \right) - TVaR \left( G|_S \right), & \text{if } j \notin S \\
1 - \frac{1}{d_i(G|_S)} + \sum_{k \in N_i(G|_S \setminus \{j\})} \frac{1}{d_k(G|_S)} [d_k(G|_S) + 1], & \text{if } j \in S
\end{cases} \\
\end{align*}
\]

so

\[
\begin{align*}
\left[ TVaR \left( G|_{S \setminus \{i\}} \right) - TVaR \left( G|_S \right) \right] &= \mathbb{1} \left\{ j \in S \right\} \cdot \left[ \frac{1}{d_i(G|_S)} \left[ \frac{1}{d_i(G|_S)} + 1 \right] + \frac{1}{d_j(G|_S)} \left[ \frac{1}{d_j(G|_S)} + 1 \right] \right] \\
& \geq \mathbb{1} \left\{ j \in S \right\} \cdot \left[ \frac{1}{d_i(G)} \left[ \frac{1}{d_i(G)} + 1 \right] + \frac{1}{d_j(G)} \left[ \frac{1}{d_j(G)} + 1 \right] \right]
\end{align*}
\]

Averaging over all possible \( S \subseteq N \), we get

\[
MV_i(G) - MV_i(G - ij) \geq \frac{1}{2} \cdot \left[ \frac{1}{d_i(G)} \left[ \frac{1}{d_i(G)} + 1 \right] + \frac{1}{d_j(G)} \left[ \frac{1}{d_j(G)} + 1 \right] \right]
\]

as

\[
\sum_{S \subseteq N} \frac{\#(S) - 1}{n!} (n - \#(S)) \mathbb{1} \{ j \in S \} = Pr \{ i \text{ arrives later than } j \} = \frac{1}{2}
\]

From the perspective of social efficiency, the link \( ij \) in \( G \) is (strictly) socially efficient if

\[
\frac{1}{d_i(G)} \left[ \frac{1}{d_i(G)} + 1 \right] + \frac{1}{d_j(G)} \left[ \frac{1}{d_j(G)} + 1 \right] > 2c.
\]

Thus we can conclude that, given any pairwise stable network \( G \) under the Myerson-value transfers, whenever a link \( ij \) is (strictly) socially efficient, it will be present in \( G \), because the increments in both \( i \)'s and \( j \)'s private benefits strictly exceed the cost of linking \( c \):

\[
MV_i(G) - MV_i(G - ij) > \frac{1}{2} \cdot 2c = c
\]
\[ MV_j(G) - MV_j(G - ij) > \frac{1}{2} \cdot 2c = c. \]

We leave a more detailed investigation of network formation in the context of risk sharing with local information constraints to future research.

6 Conclusion

This paper investigates informal risk sharing assuming that only direct neighbors can observe each others’ endowment realizations and bilateral risk-sharing arrangements can only be conditioned on commonly observed endowments. Our model generates novel predictions relative to other ways of explaining the empirical finding that in most settings informal risk sharing is imperfect. In particular, in our model centrally located individuals (households) become quasi insurance providers to more peripheral individuals. We also find that fine details of the correlation structure, in particular how correlation between endowments changes with network distance, can make a big difference in how close risk-sharing arrangements with local information constraints can get to full efficiency. This can potentially help explain why informal risk sharing works better in certain settings than in others.

While there are many theoretical extensions of the model that would be worth pursuing, such as placing the model into a dynamic context, or generalizing the monitoring structure we assumed, in future research we would also like to test some the basic assumptions (such as only direct friends’ endowment shocks influence one’s consumption level) and predictions (such as more central individuals end up with higher consumption variance) of the model empirically. This would require a dataset that has both a network component and panel data involving income shocks and consumption levels of individuals.

References


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Appendix A

Lemma 1. $T^*$ with $\langle \cdot, \cdot \rangle$ forms a Hilbert space.

Proof. We first show that $\langle \cdot, \cdot \rangle$ is a well-defined inner product. Symmetry immediately follows from the definition. Linearity in the first argument follows from the linearity of the expectation operator:

$$
\langle \alpha s + \beta t, \, r \rangle = \mathbb{E} \left[ \sum_{G_{ij} = 1} (\alpha s_{ij} + \beta t_{ij}) r_{ij} \right] = \alpha \mathbb{E} \left[ \sum_{G_{ij} = 1} s_{ij} r_{ij} \right] + \beta \mathbb{E} \left[ \sum_{G_{ij} = 1} t_{ij} r_{ij} \right] = \alpha \langle s, \, r \rangle + \beta \langle t, \, r \rangle.
$$

Positive definiteness is also obvious:

$$
\langle t, t \rangle = \mathbb{E} \left[ \sum_{G_{ij} = 1} t^2_{ij} (\omega) \right] \geq 0
$$

and $\langle t, t \rangle = 0$ if and only if $t = 0$, i.e., $t_{ij} (\omega) = 0$ for all linked $ij$ and $\mathbb{P}$-almost all $\omega \in \Omega$.

We then show that $T^*$ is a linear space. $\forall s, t \in T^*$, $\forall \alpha, \beta \in \mathbb{R}$, $\alpha s (I_{ij}) + \beta t (I_{ij})$ is also $\sigma (I_{ij})$-measurable, and

$$
\alpha s_{ij} (\omega) + \beta t_{ij} (\omega) = - (\alpha s_{ji} (\omega) + \beta t_{ji} (\omega)).
$$

Hence,

$$
\alpha s + \beta t \in T.
$$

Furthermore,

$$
\langle \alpha s + \beta t, \alpha s + \beta t \rangle = \alpha^2 \langle s, s \rangle + 2 \alpha \beta \langle s, t \rangle + \beta^2 \langle t, t \rangle < \infty,
$$

because

$$
\langle s, t \rangle^2 = \left( \sum_{ij} \mathbb{E} [s_{ij} t_{ij}] \right)^2 \leq \left( \sum_{ij} (\mathbb{E} [s_{ij}^2] \mathbb{E} [t_{ij}^2])^{\frac{1}{2}} \right)^2 \leq \sum_{ij} \mathbb{E} [s_{ij}^2] \cdot \sum_{ij} \mathbb{E} [t_{ij}^2] = \langle s, s \rangle \cdot \langle t, t \rangle < \infty.
$$

by applying the Cauchy-Schwarz inequality twice. So $\alpha s + \beta t \in T^*$.

We finally show that $T^*$ with $\langle \cdot, \cdot \rangle$ is complete. Taking $\{t^{(n)}\}$ to be a Cauchy sequence in $T^*$, then $\{t^{(n)}_{ij}\}$ must be a Cauchy sequence in a $L^2 (\mathbb{P})$ space with the inner product

$$
\langle s_{ij}, t_{ij} \rangle := \mathbb{E} [s_{ij} (\omega) t_{ij} (\omega)].
$$

As the $L^2 (\mathbb{P})$ space is complete, $t^{(n)}_{ij}$ must converge to some $t_{ij}$ in it with $\mathbb{E} [t^2_{ij} (\omega)] < \infty$. Define $t := (t_{ij})_{G_{ij} = 1}$. Clearly, $t \in T^*$, and $t^{(n)} \rightarrow t$. Hence, $T^*$ with $\langle \cdot, \cdot \rangle$ forms a Hilbert space. □
Lemma 2. The objective function in (5)

\[ J(t) := \mathbb{E} \left[ \sum_{k \in \mathcal{N}} \lambda_k u_k \left( e_k - \sum_{h \in \mathcal{N}_k} t_{kh} \right) \right] \]

is concave on \( \mathcal{T}^* \).

Proof. \( \forall s, t \in \mathcal{T}^*, \forall \alpha \in [0, 1], \)

\[
J(\alpha s + (1 - \alpha) t) \\
= \mathbb{E} \left[ \sum_i \lambda_i u_i \left( e_i - \sum_{j \in \mathcal{N}_i} (\alpha s_{ij}(\omega) + (1 - \alpha) t_{ij}(\omega)) \right) \right] \\
= \sum_i \lambda_i \mathbb{E} \left[ u_i \left( \alpha \left( e_i - \sum_{j \in \mathcal{N}_i} s_{ij}(\omega) \right) + (1 - \alpha) \left( e_i - \sum_{j \in \mathcal{N}_i} t_{ij}(\omega) \right) \right) \right] \\
\geq \sum_i \lambda_i \mathbb{E} \left[ \alpha u_i \left( e_i - \sum_{j \in \mathcal{N}_i} s_{ij}(\omega) \right) + (1 - \alpha) u_i \left( e_i - \sum_{j \in \mathcal{N}_i} t_{ij}(\omega) \right) \right] \\
= \alpha \mathbb{E} \left[ \sum_i \lambda_i u_i \left( e_i - \sum_{j \in \mathcal{N}_i} s_{ij}(\omega) \right) \right] + (1 - \alpha) \mathbb{E} \left[ \sum_i \lambda_i u_i \left( e_i - \sum_{j \in \mathcal{N}_i} t_{ij}(\omega) \right) \right] \\
= \alpha J(s) + (1 - \alpha) J(t). \]

Lemma 3. \( J \) is twice Fréchet-differentiable.
Proof. \( \forall s, t \in \mathcal{T}^* \), for \( \alpha > 0 \),

\[
J(t + \alpha s) - J(t) = \frac{\alpha}{E} \left[ \sum_i \lambda_i \left[ \frac{u_i \left( e_i - \sum_{j \in N_i} t_{ij} (\omega) - \alpha \sum_{j \in N_i} s_{ij} (\omega) \right) - u_i \left( e_i - \sum_{j \in N_i} t_{ij} (\omega) \right) }{\alpha} \right] \right]
\]

\[
= \sum_i \lambda_i \left[ u_i \left( e_i - \sum_{j \in N_i} t_{ij} (\omega) - \alpha \sum_{j \in N_i} s_{ij} (\omega) \right) - u_i \sum_{j \in N_i} s_{ij} (\omega) \right] \text{ for some } \tilde{s}_{ij} (\omega) \text{ between 0 and } \sum_{j \in N_i} s_{ij} (\omega)
\]

\[
= -\sum_i \lambda_i \left[ u_i \left( e_i - \sum_{j \in N_i} t_{ij} (\omega) - \alpha \tilde{s} (\omega) \right) \cdot \sum_{j \in N_i} s_{ij} (\omega) \right] \text{ as } \alpha \to 0
\]

\[
= -\sum_i \lambda_i \mathbb{E} \left[ u_i \left( e_i - \sum_{j \in N_i} t_{ij} (\omega) \right) \cdot \sum_{j \in N_i} s_{ij} (\omega) \right] \cdot s (\omega)
\]

\[
= \sum_i \lambda_i < f_i, s >
\]

where

\[
f_i (\omega) := -u_i \left( e_i - \sum_{j \in N_i} t_{ij} (\omega) \right) \mathbb{1}_{i \times N_i}
\]

and \( \mathbb{1}_{i \times N_i} \) is vector of 0 and 1s that equals 1 for the (directed) link \( ij \) for any \( j \in N_i \) so that

\[
\mathbb{1}_{i \times N_i} \cdot s (\omega) = \sum_{j \in N_i} s_{ij} (\omega).
\]

Define \( J' (t) : \mathcal{T}^* \to \mathbb{R} \) by

\[
J' (t) s = \sum_i \lambda_i < f_i, s >.
\]

Clearly \( J' (t) \) is a linear operator on \( \mathcal{T}^* \), and is thus the first-order Fréchet-derivative of \( J \).
Similarly, $\forall t, v, w \in T^*$,

\[
J' (t + \alpha w) v - J' (t) v = \sum_i \lambda_i \mathbb{E} \left[ \frac{\alpha}{\alpha} u_i' \left( e_i - \sum_{j \in N_i} t_{ij} (\omega) - \alpha \sum_{j \in N_i} w_{ij} (\omega) \right) \cdot \sum_{j \in N_i} v_{ij} (\omega) \right] - \sum_i \lambda_i \mathbb{E} \left[ u_i' \left( e_i - \sum_{j \in N_i} t_{ij} (\omega) \right) \cdot \sum_{j \in N_i} v_{ij} (\omega) \right]
\]

\[
= \mathbb{E} \left[ \sum_i \lambda_i \left[ u_i'' \left( e_i - \sum_{j \in N_i} t_{ij} (\omega) - \alpha \tilde{w} (\omega) \right) \cdot \alpha \sum_{j \in N_i} w_{ij} (\omega) \right] \cdot \sum_{j \in N_i} v_{ij} (\omega) \right]
\]

for some $\tilde{w}(\omega)$ between 0 and $\sum_{j \in N_i} w_{ij} (\omega)$

\[
\rightarrow \mathbb{E} \left[ \sum_i \lambda_i \left[ u_i'' \left( e_i - \sum_{j \in N_i} t_{ij} (\omega) \right) \sum_{j \in N_i} w_{ij} (\omega) \cdot \sum_{j \in N_i} v_{ij} (\omega) \right] \right] \quad \text{as } \alpha \to 0
\]

\[
= \sum_i \lambda_i E \left[ u_i'' \left( e_i - \sum_{j \in N_i} t_{ij} (\omega) \right) (1_{i \times N_i} \cdot w (\omega)) (1_{i \times N_i} \cdot v (\omega)) \right]
\]

which is clearly bilinear in $v$ and $w$, so $J$ is twice Fréchet-differentiable. \hfill \square

**Lemma 4.** For any $t \in T^*$ that solves (6), we have

\[
J' (t) = 0.
\]

**Proof.** To solve (6)

\[
\max_{i \in \mathbb{R}} J^{(ij, I_{ij})} (\tilde{t}_{ij}) := \mathbb{E} \left[ \sum_{k \in N} \lambda_k u_k \left( e_k - \sum_{h \in N_k} t_{kh} \right) \right] I_{ij}
\]

we first notice the objective function $J^{(ij, I_{ij})} (\tilde{t}_{ij})$ is strictly concave in $\tilde{t}_{ij}$ on $\mathbb{R}$. Hence, the sufficient and necessary condition for optimality is given by the FOC:

\[
\mathbb{E} \left[ \lambda_i u_i' \left( e_i - \sum_{h \in N_i} t_{ih} (\omega) \right) I_{ij} \right] = \mathbb{E} \left[ \lambda_j u_j' \left( e_j - \sum_{h \in N_j} t_{jh} (\omega) \right) I_{ij} \right]
\]

Then, $\forall s \in T^*$,

\[
J' (t) s = -\mathbb{E} \left[ \sum_i \lambda_i \left[ u_i' \left( e_i - \sum_{j \in N_i} t_{ij} (\omega) \right) \cdot \sum_{j \in N_i} s_{ij} (\omega) \right] \right]
\]

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\[- \sum_{G_{ij}=1} E \left[ \left( \lambda_i u'_i \left( e_i - \sum_{h \in N_i} t_{ih} (\omega) \right) \right) \cdot s_{ij} (\omega) \right] \]

\[- \sum_i \lambda_i \sum_{j \in N_i} E \left[ s_{ij} (I_{ij}) \cdot E \left[ \lambda_i u'_i \left( e_i - \sum_{h \in N_i} t_{ih} (\omega) \right) \cdot s_{ij} (\omega) \right] \right] \]

\[- \sum_i \lambda_i \sum_{j \in N_i} E \left[ s_{ij} (I_{ij}) \cdot 0 \right] \]

= 0.

Hence \( J' (t) = 0 \).

\[\square\]

**Proof of Proposition 1**

**Proof.** We first prove the “only if” part. Note that, given any \( t \in \mathcal{T}^* \), \( \forall i, j \),

\[ E \left[ \sum_{k \in N} \lambda_k u_k \left( e_k - \sum_{h \in N_k} t_{kh} \right) \right] = E \left[ \sum_{k \in N} \lambda_k u_k \left( e_k - \sum_{h \in N_k} t_{kh} \right) \right] \]

\[ \leq E \left[ \max_{t_{ij} \in \mathbb{R}} \sum_{k \in N} \lambda_k u_k \left( e_k - \sum_{h \in N_k} t_{kh} \right) \right] \]

This is because, conditional on \( I_{ij} \), \( t_{ij} \) must be constant across all possible states, and thus the maximization of the conditional expectation is to solve for the optimal real number \( t_{ij} \). For \( t \) to be a solution for problem (5), suppose there exists linked \( ij \) such that \( t_{ij} \) does not solve the problem (6). Then, by the inequality above, there exists another \( t_{ij} \), specified for each different realization of \( I_{ij} \) and hence each possible state of nature, that leads to higher value of \( E \left[ \sum_{k \in N} \lambda_k u_k \left( e_k - \sum_{h \in N_k} t_{kh} \right) \right] \), contradicting with the optimality of \( t \) for problem (5). Note that the “\( \mathbb{P}\text{-almost-all} \)” quantifier applies here.

For the “if” part, notice that by Lemma 4, \( t \) solves all (6) simultaneously implies that \( J' (t) = 0 \).

As \( \mathcal{T}^* \) is a Hilbert space by Lemma 1 and \( J : \mathcal{T}^* \to \mathbb{R} \) is concave by Lemma 2 and twice Fréchet-differentiable by Lemma 3, we can apply a mathematical result\(^{24}\) on convex optimization in Hilbert space asserting that, if \( J' (t) = 0 \), then \( J (t) \) is a local maximum. Moreover, this local maximum must also be a global maximum. Suppose not. Then there exists \( s \in \mathcal{T}^* \) s.t. \( J' (s) = 0 \) and \( J (s) > J (t) \). Then by the concavity of the objective function,

\[ J (\alpha s + (1 - \alpha) t) \geq \alpha J (s) + (1 - \alpha) J (t) \]

for all \( \alpha \in (0, 1) \), contradicting with the fact that \( J (t) \) is a local maximum. \( \square \)

**Lemma 5.** The set of profiles of final consumption induced by the profiles of transfer rules \( t \) in \( \mathcal{T}^* \) is convex.

\(^{24}\)See, for example, Theorem 30.2.2.(b) in Blanchard and Brüning (2012) on pp.394-395.
Proof. Let $x, x'$ be two profiles of allocation rules induced by $t, t'$ respectively. Then $\forall \lambda \in [0, 1],$

$$\lambda x_i(\omega) + (1 - \lambda) x'_i(\omega) = \lambda \left[ e_i - \sum_{j \in N_i} t_{ij}(\omega) \right] + (1 - \lambda) \left[ e_i - \sum_{j \in N_i} t'_{ij}(\omega) \right]$$

$$= e_i - \sum_{j \in N_i} \left[ \lambda t_{ij}(\omega) + (1 - \lambda) t'_{ij}(\omega) \right]$$

Thus $(\lambda x + (1 - \lambda) x')$ can be induced by $(\lambda t + (1 - \lambda) t').$ $T^*$, as a Hilbert space, is convex, so the set of profiles of allocations rules induced by the profiles of transfer rules in $T^*$ must also be convex.

Proof of Proposition 2

Proof. Following the proof of Lemma 2, we can easily show, by the strict concavity of $u_i(\cdot)$, that the objective function in (5) is strictly concave in the profile of final consumption $x$. Lemma 5 shows that the set of admissible profiles of final consumptions induced by the set of transfer rules in $T^*$ is convex. Hence, there is at most of one profile of allocation rules that solves (5).

Lemma 6. Given any real vector $c \in \mathbb{R}^n$ such that $\sum_{i \in N} c_i = 0$, there exists a real vector $\mu \in \mathbb{R}^{\sum_i d_i}$ such that $\mu_{ik} + \mu_{ki} = 0$ for every linked pair $ik$ and

$$\sum_{k \in N_i} \mu_{ik} = c_i.$$ (19)

The solution is unique if and only if the network is minimally connected.

Proof. With the restrictions that $\mu_{ik} = -\mu_{ki}$ for all linked pair $ik$, (19) constitutes a system of $n$ linear equations with $\frac{1}{2} \sum_{i \in N} d_i$ variables $\mu_{ik}$. Summing up all the $n$ equations, we have

$$0 = \sum_{i \in N, \ G_{ij} = 1} (\mu_{ik} + \mu_{ki}) = \sum_{i \in N} c_i = 0.$$ 

Hence, the $n$ linear equations impose at most $(n - 1)$ linearly independent conditions.

Viewing (19) in vector form,

$$C \mu = c$$

where $C$ is a $n \times \frac{1}{2} \sum_{i \in N} d_i$ matrix. Note that in each column of $C$, denoted $C_{ij}$ for $i < j$, there are either no nonzero entries (when $G_{ij} = 0$), or just two nonzero entries: 1 on the $i$-th row and −1 on the $j$-th row when $G_{ij} = 1$. Suppose $G_{ij} = 1$. Then, given any subset of agents $S$ that include $i$ and $j$, if the rows of $C$ corresponding to $S$ are linearly dependent, these rows must sum to 0: this can be true only if all entries $ik$ with $i \in S$ and $k \notin S$ are zero, implying that $S$ form a component under $G$, and thus $G$ is not connected if $\#(S) < n$. This is in contradiction with the supposition that $G$ is connected when $\#(S) < n$. Hence, $C$ must have exactly $(n - 1)$ linearly independent rows.
Let \( \tilde{C} \) and \( \tilde{c} \) be the first \((n-1)\) rows of \( C \) and \( c \). Then, as \( \tilde{C} \) has full row rank, there always exists a solution to \( \tilde{C} \mu = \tilde{c} \), and any of the solutions \( \mu \) must also solve the equation \( C \mu = c \). The solution is unique if and only if the component is minimally connected, when there are precisely \((n-1)\) links and thus \( \tilde{C} \) is an invertible square matrix.

We can obtain one particular solution using the following algorithm. First, we can arbitrarily select a subset of links that minimally connect the nodes, i.e., the graph restricted to this subset of links is minimally connected. Then, there must exist at least one peripheral node, and we can first easily obtain \( \mu_{i,j} \) for all such peripheral nodes \( i \in P_1 := \{k \in N: d_k = 1\} \). Then, we can look for new peripheral nodes ignoring the links involving nodes in \( P_1 \), and obtain \( \mu_{i,j} \) for all \( i \in P_2 := \{k \in N: k \notin P_1 \land G_{kj} = 1 \text{ for some } j \in P_1\} \) with all previously calculated \( \mu \)'s taken as given. We iterate this process until we exhaust all nodes. Then we are left with a profile of \( \mu \) that solves (19).

\[ \square \]

**Lemma 7.** A linear profile of transfer rules \( t = (\alpha, \beta, \mu) \) is Pareto efficient if \( \forall i,j \) s.t. \( G_{ij} = 1 \),

\[
\begin{align*}
\alpha_{ij} &= \frac{1}{2} \left( 1 - \sum_{k \in N \setminus \{j\}} \alpha_{ik} + \sum_{k \in N_{ij}} \beta_{jki} + \gamma_{ij} \right) \\
\beta_{ijk} &= \frac{1}{2} \left[ \alpha_{ki} - \alpha_{kj} + \sum_{h \in N_{ijk}} (\beta_{ikh} - \beta_{jkh}) \right. \\
&\left. - \sum_{h \in N_{ik} \setminus \{j\}} \beta_{ikh} + \sum_{h \in N_{jk} \setminus \{i\}} \beta_{jkh} + \gamma_{ij} \right] \quad \forall k \in N_{ij} \\
\gamma_{ij} &= \frac{p}{1 + (d_{ij} + 1)p} \left[ \sum_{k \in N_i \setminus \{j\}} \left( \alpha_{ki} - \sum_{h \in N_{ik} \setminus \{j\}} \beta_{ikh} \right) \right. \\
&\left. \sum_{k \in N_j \setminus \{i\}} \left( \alpha_{kj} - \sum_{h \in N_{jk} \setminus \{i\}} \beta_{jkh} \right) \right. \\
&\left. - \sum_{k \in N_{ij}} \left( \sum_{h \in N_{ik} \setminus \{j\}} \beta_{ikh} - \sum_{h \in N_{jk} \setminus \{i\}} \beta_{jkh} \right) \right]
\end{align*}
\]

**Proof.** For each \( k \in N_i \setminus \{j\} \), we then have

\[
\sum_{k \in N_i \setminus \{j\}} t_{ik} = e_i \sum_{k \in N_i \setminus \{j\}} \alpha_{ik} - \sum_{k \in N_i \setminus \{j\}} \alpha_{ki}e_k + \sum_{k \in N_i \setminus \{j\}} \sum_{h \in N_{ik}} \beta_{ikh}e_h + c_{ij}
\]

\[
= e_i \sum_{k \in N_i \setminus \{j\}} \alpha_{ik} - \sum_{k \in N_i \setminus \{j\}} \alpha_{ki}e_k + \sum_{k \in N_i \setminus \{j\}} \left( \beta_{ikj}e_j + \sum_{h \in N_{ijk}} \beta_{ikh}e_h \right) + \sum_{k \in N_i \setminus \{j\}} \sum_{h \in N_{ijk}} \beta_{ikh}e_h
\]

so that

\[
t_{ij} = \frac{1}{2} e_i - \frac{1}{2} e_j - \frac{1}{2} \sum_{k \in N_i \setminus \{j\}} \alpha_{ik}e_i + \frac{1}{2} \sum_{k \in N_j \setminus \{i\}} \alpha_{jki}e_j - \frac{1}{2} \sum_{k \in N_{ij}} \left( \beta_{ikj}e_j - \beta_{jki}e_i \right)
\]

\[+ \frac{1}{2} \sum_{k \in N_{ij}} \left( \alpha_{ki} - \alpha_{kj} \right) e_k - \sum_{h \in N_{ijk}} \left( \beta_{ikh} - \beta_{jkh} \right) e_h \]

\[- \frac{1}{2} \sum_{k \in N_i \setminus \{j\}} \sum_{h \in N_{ijk}} \beta_{ikh}e_h + \frac{1}{2} \sum_{k \in N_j \setminus \{i\}} \sum_{h \in N_{ijk}} \beta_{jkh}e_h
\]

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Proof.
We begin by proving the first part, which only involves triads. We rewrite \( (16) \) in the
Proof of Proposition 4
The last equality is obtained by collecting terms and switching summand indice.

\[
- \frac{1}{2(1 + (d_{ij} + 1) \rho)} \left( e_i + e_j + \sum_{k \in N_{ij}} e_k \right) \sum_{k \in N_{ij}} \left( \sum_{h \in N_{ik} \setminus \bar{N}_j} \beta_{ikh} - \sum_{h \in N_{jk} \setminus \bar{N}_i} \beta_{jkh} \right) \\
+ \frac{1}{2(1 + (d_{ij} + 1) \rho)} \left( e_i + e_j + \sum_{k \in N_{ij}} e_k \right) \sum_{k \in N_{ij} \setminus \bar{N}_i} \left( \alpha_{ki} - \sum_{h \in N_{ik} \setminus \bar{N}_j} \beta_{ikh} \right) \\
- \frac{1}{2(1 + (d_{ij} + 1) \rho)} \left( e_i + e_j + \sum_{k \in N_{ij}} e_k \right) \sum_{k \in N_{ij} \setminus \bar{N}_i} \left( \alpha_{kj} - \sum_{h \in N_{jk} \setminus \bar{N}_i} \beta_{jkh} \right) \\
= \frac{1}{2} \left\{ 1 - \sum_{k \in N_i \setminus \{i\}} \alpha_{ik} + \sum_{k \in N_{ij}} \beta_{jkh} \right\} \cdot e_i \\
- \sum_{k \in N_j \setminus \{j\}} \left( \alpha_{kj} - \sum_{h \in N_{jk} \setminus \bar{N}_i} \beta_{jkh} \right) \cdot e_j \\
+ \frac{1}{2} \sum_{k \in N_{ij}} \left\{ \alpha_{ki} - \alpha_{kj} + \sum_{h \in N_{ihk}} (\beta_{ikh} - \beta_{jkh}) - \sum_{h \in N_{ihk} \setminus \bar{N}_j} \beta_{ikh} + \sum_{h \in N_{jk} \setminus \bar{N}_i} \beta_{jkh} \right\} \cdot e_k + C_{ij}
\]

The last equality is obtained by collecting terms and switching summand indice. \( \Box \)

**Proof of Proposition 4**

*Proof.* We begin by proving the first part, which only involves triads. We rewrite \((16)\) in the following way:
\[
\begin{cases}
2\alpha_{ij} + \sum_{k \in N_i \setminus \{j\}} \alpha_{ik} - \gamma_{ij} = 1, & \forall G_{ij} = 1 \\
\alpha_{ki} - \alpha_{kj} + \gamma_{ij} = 0 & \forall k \in N_{ij}, \ \forall G_{ij} = 1; \\
\gamma_{ij} = \frac{\rho}{1 + (d_{ij} + 1) \rho} \left( \sum_{k \in N_i \setminus N_j} \alpha_{ki} - \sum_{k \in N_j \setminus N_i} \alpha_{kj} \right), & \forall G_{ij} = 1;
\end{cases}
\]

In matrix form we write

\[
\begin{bmatrix}
\tilde{A} \\
\tilde{M}
\end{bmatrix}
\begin{pmatrix}
\alpha \\
\gamma
\end{pmatrix}
= 
\begin{pmatrix}
\tilde{b} \\
0
\end{pmatrix}
\begin{align*}
\text{(1)} \land \text{(3)} \\
\text{(2)}
\end{align*}
\]

where \(\alpha, \gamma\) are both \(\sum_i d_i\)-dimensional vectors, \(\tilde{A}\) is a \((2 \sum_i d_i) \times (2 \sum_i d_i)\) square matrix, \(\tilde{b} := \left( \begin{array}{c}
\frac{1}{\sum_i d_i} \\
0
\end{array} \right)\) is a \((2 \sum_i d_i)\)-dimensional vector, \(\tilde{M}\) is a \((\sum_{G_{ij}=1} d_{ij}) \times (2 \sum_i d_i)\) rectangular matrix, and \(0\) is a \((\sum_{G_{ij}=1} d_{ij})\)-dimensional vector. The upper block \(\tilde{A} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \tilde{b}\) corresponds to equations in (1) and (3), while the lower block \(\tilde{M} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = 0\) corresponds to equations in (2).

When \(A\) is invertible, there is a unique solution in \((\alpha', \gamma')\)' that solves (1) and (3), so all the \(2 \sum_i d_i\) equations corresponding to (1) and (3) must be linearly independent, and thus \(\tilde{A}\) is also invertible.

We now show that the equations in (2) cannot be linearly independent from those in (1) and (3). In other words, there exists a nonzero vector \(\xi \in \mathbb{R}^{2 \sum_i d_i + \sum_{G_{ij}=1} d_{ij}}\) such that

\[
\xi' \begin{bmatrix}
\tilde{A} \\
\tilde{M}
\end{bmatrix} \begin{pmatrix}
\tilde{b} \\
0
\end{pmatrix} = (0, 0, \ldots, 0)^{2 \sum_i d_i + 1}.
\]

For any linked pair \(ij\), multiply (3), \(ij\) (the \(ij\)-th equation in (3)) with \((1 + (d_{ij} + 1) \rho)\), obtaining

\[
[1 + (d_{ij} + 1) \rho] \gamma_{ij} = \rho \left( \sum_{h \in N_i \setminus N_j} \alpha_{hi} - \sum_{h \in N_j \setminus N_i} \alpha_{hj} \right)
\]

which is equivalent to

\[
[1 + (d_{ij} + 1) \rho] \gamma_{ij} = \rho \left( \sum_{h \in N_i} \alpha_{hi} - \sum_{h \in N_j} \alpha_{hj} - \sum_{h \in N_{ij}} (\alpha_{hi} - \alpha_{hj}) - \alpha_{ji} + \alpha_{ij} \right)
\]

Adding (2) \(ij\) for all \(h \in N_{ij}\) to (4), we get

\[
[1 + (d_{ij} + 1) \rho] \gamma_{ij} = \rho \left( \sum_{h \in N_i} \alpha_{hi} - \sum_{h \in N_j} \alpha_{hj} + d_{ij} \rho - \alpha_{ji} + \alpha_{ij} \right)
\]
which is equivalent to

$$
(1 + \rho) \gamma_{ij} = \rho \left( \sum_{h \in N_i} \alpha_{hi} - \sum_{h \in N_j} \alpha_{hj} + \alpha_{ij} - \alpha_{ji} \right) \tag{5}_{ij}
$$

Given any linked triads $ijk$, summing up (5)$_{ij}$, (5)$_{jk}$, (5)$_{ki}$, we have

$$
(1 + \rho) (\gamma_{ij} + \gamma_{jk} + \gamma_{ki}) = \rho (\alpha_{ij} + \alpha_{jk} + \alpha_{ki} - \alpha_{ji} - \alpha_{kj} - \alpha_{ik}) \tag{6}_{ijk}.
$$

Moreover, by (2)$_{ijk}$, (2)$_{jki}$, (2)$_{kij}$, we have

$$
\alpha_{ij} - \alpha_{ik} + \alpha_{jk} - \alpha_{ji} + \alpha_{ki} - \alpha_{kj} = -\gamma_{jk} - \gamma_{ki} - \gamma_{ij} \tag{7}_{ijk}.
$$

Hence, (6) $- \rho \times (7)$ gives

$$
(1 + 2\rho) (\gamma_{ij} + \gamma_{jk} + \gamma_{ki}) = 0.
$$

For $n \geq 4$, we have $1 + 2\rho > 0$ and thus

$$
\gamma_{ij} + \gamma_{jk} + \gamma_{ki} = 0 \tag{8}_{ijk}
$$

Taking (1)$_{ki} - (1)$_{kj} + (2)$_{ijk}$, we obtain

$$
\gamma_{ij} - \gamma_{kj} + \gamma_{ki} = 0
$$

By (3)$_{jk} + (3)$_{kj}$, we have $\gamma_{jk} + \gamma_{kj} = 0$ and thus

$$
\gamma_{ij} + \gamma_{jk} + \gamma_{ki} = 0 \tag{9}_{ijk}
$$

Then (8)$_{ijk}$ $- (9)$_{ijk}$ leads to $0 = 0$. But notice that the vector $\xi \in \mathbb{R}^{2\sum_i d_i + \sum_{G_{ij}=1} d_{ij}}$ that characterizes all the row operations conducted above must be nonzero. In particular, we must have

$$
\xi (1)$_{ki} \neq 0, \quad \xi (1)$_{kj} \neq 0,
$$

because (1)$_{ki}$, (1)$_{kj}$ are used to obtain (9)$_{ijk}$ and nowhere else. As $\tilde{A}$ is of full row rank, this leads to the conclusion that (2) must be a linear combination of (1) and (3).

We now prove the second part, the statement for cycles of any size. Note that we still have

$$
(1 + \rho) \gamma_{ij} = \rho \left( \sum_{h \in N_i} \alpha_{hi} - \sum_{h \in N_j} \alpha_{hj} + \alpha_{ij} - \alpha_{ji} \right) \tag{5}_{ij}
$$

Given any cycle $i_1i_2...i_mi_1$, summing up (5)$_{i_1i_2}$, (5)$_{i_2i_3}$, ..., (5)$_{i_mi_1}$, we have

$$
(1 + \rho) (\gamma_{i_1i_2} + \gamma_{i_2i_3} + ... + \gamma_{i_mi_1}) = \rho (\alpha_{i_1i_2} + ... + \alpha_{i_{m-1}i_m} - \alpha_{i_{m-2}i_{m-1}} - ... - \alpha_{i_1i_m}) \tag{10}
$$

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By \( \frac{1}{i_{1i_2}} - \frac{1}{i_{2i_1}} \) and \( \gamma_{ij} + \gamma_{ji} = 0 \), we have
\[
\alpha_{i_{1i_2}} - \alpha_{i_{2i_1}} = \sum_{h \in N_{i_2}} \alpha_{i_2h} - \sum_{h \in N_{i_1}} \alpha_{i_1h} + 2\gamma_{i_{1i_2}}.
\]
Summing over \( i_{1i_2}, ..., i_mi_1 \),
\[
\alpha_{i_{1i_2}} + ... + \alpha_{i_mi_1} - \alpha_{i_{2i_1}} - ... - \alpha_{i_{1i_m}} = 2(\gamma_{i_{1i_2}} + \gamma_{i_{2i_3}} + ... + \gamma_{i_{m_i_1}}) \tag{11}
\]
Then \( 10 + \rho \times 11 \) gives
\[
(1 - \rho)(\gamma_{i_{1i_2}} + \gamma_{i_{2i_3}} + ... + \gamma_{i_{m_i_1}}) = 0.
\]
For \( \rho < 1 \), we have
\[
\gamma_{i_{1i_2}} + \gamma_{i_{2i_3}} + ... + \gamma_{i_{m_i_1}} = 0.
\]

**Lemma 8.** In any network, \( \forall \epsilon \in (0, 1), \alpha(\rho) = A(\rho)^{-1} \mathbf{1} \) is uniformly bounded on \( [0, 1 - \epsilon] \setminus \Lambda_0 \).

**Proof.** Suppose not. Then there exists \( \epsilon \in (0, 1) \) and a sequence \( \{\rho_m\} \subseteq [0, 1 - \epsilon] \setminus \Lambda_0 \) such that \( \rho_m \to \rho \in [0, 1 - \epsilon] \) and \( |\alpha_{ij}(\rho_m)| \to \infty \) for some \( ij \) (This follows from the continuity of \( \alpha(\rho) \) on \( [0, 1 - \epsilon] \setminus \Lambda_0 \)). Note that
\[
x_j = \alpha_{jj}e_j + \alpha_{ij}e_i + \sum_{k \in N \setminus \{i\}} \alpha_{kj}e_k
\]
where
\[
\alpha_{jj} := 1 - \sum_{k \in N} \alpha_{jk}.
\]
Then, suppressing \( \sigma^2 \) to simplify notations,
\[
Var(x_j) = \alpha_{jj}^2 + \alpha_{ij}^2 + \sum_{k \in N \setminus \{i\}} \alpha_{kj}^2 + 2\rho \left( \alpha_{jj} \alpha_{ij} + \sum_{k \in N \setminus \{i\}} \alpha_{kj} (\alpha_{jj} + \alpha_{ji}) + \sum_{h,k \in N \setminus \{i\}} \alpha_{hj} \alpha_{kj} \right)
\]
\[
= (1 - \rho) \left( \alpha_{jj}^2 + \alpha_{ij}^2 + \sum_{k \in N \setminus \{i\}} \alpha_{kj}^2 \right) + \rho \left( \alpha_{jj} + \alpha_{ij} + \sum_{k \in N \setminus \{i\}} \alpha_{kj} \right)^2
\]
\[
\geq \epsilon \left( \alpha_{jj}^2 + \alpha_{ij}^2 + \sum_{k \in N \setminus \{i\}} \alpha_{kj}^2 \right) + \rho \left( \alpha_{jj} + \alpha_{ij} + \sum_{k \in N \setminus \{i\}} \alpha_{kj} \right)^2.
\]
Notice that the second term is nonnegative. Hence, \( |\alpha_{ij}(\rho_m)| \to \infty \) implies that \( Var(x_j(\rho_m)) \to \infty \).

This violates the Pareto efficiency of \( \alpha(\rho_m) \), which can be seen from the following. Consider the autarky transfer \( \alpha \equiv 0 \), which leads to a consumption variance of \( \sigma^2 \) for each individual. Let \( TV(\alpha) \) denote the sum of each individual’s consumption variances under the transfer.

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α and the induced corresponding state-independent transfers \( c(\alpha) \). Then, \( TV(0) = n\sigma^2 \) and \( TV(\alpha(\rho_m)) > n\sigma^2 \) for some large enough \( m \). Define \( \Delta := TV(\alpha(\rho_m)) - n\sigma^2 > 0 \) and

\[
\tilde{c}_k := c_k(\alpha(\rho_m)) + \frac{1}{2} r \left( \sigma^2 - Var\left[x_k(\alpha(\rho_m))\right] + \frac{1}{n}\Delta \right)
\]

so

\[
CE_k(0) = c_k(\alpha(\rho_m)) + \frac{1}{2} r \left( \sigma^2 - Var\left[x_k(\alpha(\rho_m))\right] + \frac{1}{n}\Delta \right) - \frac{1}{2} r\sigma^2
\]

\[
= CE_k(x_k(\alpha(\rho_m))) + \frac{1}{2} r\Delta
\]

\[
> CE_k(x_k(\alpha(\rho_m)))
\]

i.e., each individual \( k \) obtains a strictly higher certainty equivalent. Moreover, notice that

\[
\sum_{k \in N} \tilde{c}_k = \sum_{k \in N} c_k(t) + \frac{1}{2} r (n\sigma^2 - TV(\alpha(\rho_m)) + \Delta) = 0.
\]

By Lemma 6, \( (\tilde{c}_k) \) can be achieved by a profile of state-independent transfers \( (\mu_{ij}) \) in the network. Hence, we can find a profile of transfer rules in \( \mathcal{T}^* \) that achieves a strict Pareto improvement over \( \alpha(\rho_m) \), contradicting the Pareto efficiency of \( \alpha(\rho_m) \).

**Lemma 9.** In any network, \( \forall \epsilon \in (0, 1), \alpha(\rho) = A(\rho)^{-1} 1 \) is uniformly bounded on \( [-\frac{\epsilon}{n-1}, 0] \setminus \Lambda_0 \).

**Proof.** For any \( \rho \geq -\epsilon \), suppressing \( \sigma^2 \), the variance-covariance matrix

\[
\Sigma(\rho) = \begin{pmatrix}
1 & \rho & \cdots & \rho \\
\rho & 1 & \cdots & \rho \\
\vdots & \vdots & \ddots & \vdots \\
\rho & \rho & \rho & 1
\end{pmatrix}
\]

\[
= -\rho \cdot \begin{pmatrix}
1 \quad -1 \quad \cdots \quad -1 \\
\cdots \quad \cdots \quad \cdots \quad \cdots \\
-1 \quad -1 \quad \cdots \quad n-1
\end{pmatrix} + (1 + (n-1)\rho) \cdot I
\]

\[
= |\rho| L + (1 - (n-1)|\rho|) \cdot I.
\]

Notice that the \( L \) matrix is the Laplacian matrix associated with the complete network. Note that \( L \) has two distinct eigenvalues: \( 0 \) with multiplicity \( 1 \) and \( n \) with multiplicity \( n-1 \). Given any nonzero vector \( z \in \mathbb{R}^n \),

\[
0 = \min_{z \in \mathbb{R}^n \setminus \{0\}} \frac{z^tLz}{z^t z}
\]

and thus

\[
1 - (n-1)|\rho| = \min_{z \in \mathbb{R}^n \setminus \{0\}} \frac{z^t\Sigma(\rho)z}{z^t z}.
\]
For each individual $j$, denote $z_j = (\alpha_{1j}, \ldots, \alpha_{nj})$ with $\alpha_{jj} := 1 - \sum_{k \in N_i} \alpha_{jk}$ and $\alpha_{kj} = 0$ if $G_{kj} = 0$. Then,

$$Var(x_j) = z_j^\prime \Sigma(\rho) z_j \geq (1 - (n - 1) |\rho|) \cdot z_j^\prime z_j \geq (1 - \epsilon) \cdot z_j^\prime z_j.$$  

Hence, $\forall \rho \in [-\frac{\epsilon}{n-1}, 0] \setminus \Lambda_0$, let $z_j(\rho)$ be constructed from the Pareto efficient $\alpha(\rho) = A(\rho)^{-1} 1$. Then we must have

$$\|z_j\|^2 \leq \frac{n}{1 - \epsilon}.$$  

Hence, $\alpha(\rho)$ is uniformly bounded on $[-\frac{\epsilon}{n-1}, 0] \setminus \Lambda_0$. \hfill \square

**Proof of Proposition 5.**

**Proof.** On $\Lambda \setminus \Lambda_0$, $A(\rho)$ is invertible and thus $\alpha = A(\rho)^{-1} 1$ constitutes a solution to (17), which defines a linear, strictly bilateral and Pareto efficient profile of transfer rules.

We now deal with possible $\rho$ in $\Lambda_0$. Fix a small $\epsilon > 0$. By Lemma 8 and Lemma 9, $\alpha(\rho)$ is uniformly bounded on $[-\frac{1}{n-1} + \epsilon, 1 - \epsilon]$. Then $\alpha(\rho)$ is contained in a compact set. Take any $\{\rho_n\} \subseteq \Lambda \setminus \Lambda_0$ s.t. $\rho_n \to \rho$, then there exists an accumulation point in the compact set $\alpha^*$ s.t.

$$\alpha(\rho_n) \to \alpha^*.$$  

Then by continuity of $A(\rho)$ given a network structure, $\max_{\rho \in [0,1]} \|A(\rho)\|$ exists and it is finite, hence we have

$$\|A(\rho) \alpha^* - 1\| = \|A(\rho) \alpha^* - A(\rho_n) \alpha(\rho_n)\| \leq \|A(\rho) - A(\rho_n)\| \|\alpha^*\| + \|A(\rho_n)\| \|\alpha^* - \alpha(\rho_n)\| \to 0.$$  

Furthermore, notice that $\gamma(\rho_n) = \Gamma(\rho_n) \alpha(\rho_n) \to \gamma(\rho) = \Gamma(\rho) \alpha^*$. As $\gamma(\rho_n)$ satisfies the property in Proposition 4 ("Kirchoff’s Voltage Law"), its limit $\gamma(\rho)$ must also satisfy it.

Hence, for any $\rho \in \Lambda_0 \cap (-\frac{1}{n-1}, 1)$, there exists $\epsilon > 0$ s.t. $\rho \in (-\frac{1}{n-1} + \epsilon, 1 - \epsilon)$, so the existence of solution at $\rho$ is established by the above. The only possibilities that have not been accounted for is the cases when $A(\rho)$ is singular at $\rho = -\frac{1}{n-1}$ or 1.

At $\rho = 1$, $\alpha^* \equiv 0$ is clearly a Pareto efficient transfer rule. In fact, any transfer $\alpha$ that satisfies the “Kirchhoff Circuit Law” (KCL)

$$\sum_{j \in N_i} (\alpha_{ij} - \alpha_{ji}) = 0 \quad \forall i \in N$$

is Pareto efficient.
At \( \rho = -\frac{1}{n-1} \), if there is an individual \( i \) with \( d_i = n - 1 \), i.e., \( i \) is linked to everyone else in \( N \), then, the transfer characterized by \( \alpha \) such that

\[
\alpha_{ij} = 0, \quad \alpha_{ji} = 1, \quad \forall j \in N \setminus \{i\}
\]

and

\[
\alpha_{jk} = 0 \quad \forall j, k \in N \setminus \{i\}
\]

is Pareto efficient, as it achieves zero total variance (because \( \sum_i e_i(\omega) \equiv 0 \)). If there is no individual that is linked to everyone else, consider any individual \( j \). As \( d_j < n - 1 \), there exists some \( k \in N \) s.t. \( G_{jk} = 0 \) and thus \( z_{jk} = 0 \). Let \( P_1 = \mathbf{1}' (\mathbf{1}' \mathbf{1})^{-1} \mathbf{1}' \) and \( Q_1 = \mathbf{I} - P_1 \). As \( \mathbf{1} \) is the eigenvector associated with the smallest eigenvalue 0 for the Laplacian matrix \( L \), and all other eigenvalues are \( n \), it follows that

\[
z_j' L z_j = n \cdot z_j' Q_1 z_j.
\]

Consider a sequence \( \{\rho_n\} \subseteq (-\frac{1}{n-1}, -\frac{1}{n-1} + \epsilon) \) such that \( \rho_n \to -\frac{1}{n-1} \), and let \( z(\rho_n) \) be the Pareto efficient final consumption profile constructed from \( \alpha(\rho_n) \).

\[
\begin{align*}
\text{Var} (x_j (\rho_n)) &= z_j' (\rho_n) \Sigma (\rho_n) z_j (\rho_n) \\
&= |\rho_n| z_j' (\rho_n) L z_j (\rho_n) + (1 - (n - 1)|\rho_n|) \cdot \|z_j (\rho_n)\|^2 \\
&= n |\rho_n| z_j' (\rho_n) Q_1 z_j (\rho_n) + (1 - (n - 1)|\rho_n|) \cdot \|z_j (\rho_n)\|^2 \\
&= n |\rho_n| \|Q_1 z_j (\rho_n)\|^2 + (1 - (n - 1)|\rho_n|) \cdot \|z_j (\rho_n)\|^2 \\
&\geq \|Q_1 z_j (\rho_n)\|^2 = \|z_j (\rho_n)\|^2 - \frac{1}{n} |\mathbf{1}' z_j (\rho_n)|^2
\end{align*}
\]

Suppose that \( \|z_j (\rho_n)\| \to \infty \). Notice that

\[
Q_1 z_j (\rho_n) = z_j (\rho_n) - \frac{\mathbf{1}' z_j (\rho_n)}{n} \cdot \mathbf{1}
\]

If \( |\mathbf{1}' z_j (\rho_n)| \) is bounded, then clearly

\[
\|Q_1 z_j (\rho_n)\|^2 = \|z_j (\rho_n)\|^2 - \frac{1}{n} |\mathbf{1}' z_j (\rho_n)|^2 \to \infty.
\]

If otherwise \( |\mathbf{1}' z_j (\rho_n)| \to \infty \), then consider the \( k \)-th element of \( Q_1 z_j (\rho_n) \) (recall that \( G_{jk} = 0 \)). As \( z_{jk} = 0 \), we have

\[
[Q_1 z_j (\rho_n)]_k = -\frac{1}{n} \mathbf{1}' z_j (\rho_n)
\]

so

\[
\|Q_1 z_j (\rho_n)\|^2 \geq \frac{1}{n^2} |\mathbf{1}' z_j (\rho_n)|^2 \to \infty.
\]

Hence, in both cases, we get

\[
\text{Var} (x_j (\rho_n)) \geq \|Q_1 z_j (\rho_n)\|^2 \to \infty,
\]

contradicting the Pareto efficiency of \( x(\rho_n) \). Hence, \( \|z(\rho_n)\| \) must be bounded, which implies that \( \|\alpha(\rho_n)\| \) is bounded. Hence, take any limit of a subsequence \( \rho_{n_k} \) of \( \rho_n \) such that \( \alpha (-\frac{1}{n-1}) := \lim_{k \to \infty} \alpha (\rho_{n_k}) \) is well defined, and then \( \alpha (-\frac{1}{n-1}) \) must solve system (16) at \( \rho = -\frac{1}{n-1} \) and is thus Pareto efficient. \( \square \)
Appendix B

Proposition 6.

(Uniqueness in Minimally Connected Networks) Under the independent CARA-Normal setting, if the network is minimally connected, then there is a unique profile of transfer rules in $T^*$ that is Pareto efficient, and it takes the form of the local equal sharing rule.

Proof. Consider minimally connected network $G$. For Pareto efficiency, we need for all linked pair $ij$

$$
\frac{E_{ij} [u_i (x_i)]}{E_{ij} [u_j (x_j)]} = \alpha_{ij}.
$$

As the network is minimally connected, we have $N_{ij} = \emptyset$. Notice that

$$
\mathbb{E} \left[ r e^{-r (e_i - t_{ij} - \sum_{k \in N_i \setminus \{j\}} t_{ik}(e_i,e_k))} \mid e_i, e_j \right] = \alpha_{ij} \mathbb{E} \left[ r e^{-r (e_j + t_{ij} - \sum_{h \in N_j \setminus \{i\}} t_{jh}(e_j,e_h))} \mid e_i, e_j \right].
$$

By independence,

$$
\mathbb{E} \left[ r e^{-r (e_i - t_{ij} - \sum_{k \in N_i \setminus \{j\}} t_{ik}(e_i,e_k))} \mid e_i \right] = \alpha_{ij} \mathbb{E} \left[ r e^{-r (e_j + t_{ij} - \sum_{h \in N_j \setminus \{i\}} t_{jh}(e_j,e_h))} \mid e_j \right],
$$

$$
\Leftrightarrow \frac{r}{e^{-r(e_i - t_{ij})}} \cdot \left( \prod_{k \in N_i \setminus \{j\}} \mathbb{E} \left[ e^{r t_{ik}(e_i,e_k)} \mid e_i \right] \right) = \frac{r}{e^{-r(e_j + t_{ij})}} \cdot \left( \prod_{h \in N_j \setminus \{i\}} \mathbb{E} \left[ e^{r t_{jh}(e_j,e_h)} \mid e_j \right] \right),
$$

$$
\Rightarrow e_i - t_{ij} = e_j + t_{ij} - \frac{1}{r} \sum_{k \in N_i \setminus \{j\}} \ln \mathbb{E} \left[ e^{r t_{ik}(e_i,e_k)} \mid e_i \right] = e_j + t_{ij} - \frac{1}{r} \sum_{h \in N_j \setminus \{i\}} \ln \mathbb{E} \left[ e^{r t_{jh}(e_j,e_h)} \mid e_j \right] - \frac{1}{r} \ln \alpha_{ij}
$$

$$
\Rightarrow t_{ij} = \frac{1}{2} e_i - \frac{1}{2} e_j - \frac{1}{2r} \sum_{k \in N_i \setminus \{j\}} \ln T_{ik} + \frac{1}{2r} \sum_{h \in N_j \setminus \{i\}} \ln T_{jh} + \frac{1}{2r} \ln \alpha_{ij}
$$

(20)

$$
\frac{1}{r} \ln T_{ij} = \frac{1}{2} e_i + \frac{1}{2} r \sigma^2 - \frac{1}{2} \sum_{k \in N_i \setminus \{j\}} \ln T_{ik} + \frac{1}{2r} \ln \alpha_{ij}.
$$

Then, taking conditional expectations of (20), we have

$$
T_{ij} = e^{r \left( \frac{1}{2} e_i + \frac{1}{2} r \sigma^2 - \frac{1}{2} \sum_{k \in N_i \setminus \{j\}} \ln T_{ik} + \frac{1}{2r} \ln \alpha_{ij} \right)} \cdot \mathbb{E} \left[ e^{r \left( \frac{1}{2} e_j + \frac{1}{2} r \sigma^2 - \frac{1}{2} \sum_{h \in N_j \setminus \{i\}} \ln T_{jh} + \frac{1}{2r} \ln \alpha_{ij} \right)} \mathbb{E} \left[ T_{jh}^{1/2} \right] \right],
$$

and

$$
\frac{1}{r} \ln T_{ij} = \frac{1}{2} e_i + \frac{1}{2} r \sigma^2 - \frac{1}{2} \sum_{k \in N_i \setminus \{j\}} \frac{1}{r} \ln T_{ik} + \frac{1}{2r} \ln \alpha_{ij} + \sum_{h \in N_j \setminus \{i\}} \ln \mathbb{E} \left[ T_{jh}^{1/2} \right].
$$
Introducing notation

\[ \tilde{T}_{ij} = \frac{1}{r} \ln T_{ij}, \]

we have

\[ \tilde{T}_{ij} = \frac{1}{2} e_i \frac{1}{2} \sum_{k \in N_i \setminus \{j\}} \tilde{T}_{ik} + c_{ij} \]

\[ \Rightarrow \]

\[ \sum_{j \in N_i} \tilde{T}_{ij} = \frac{d_i}{2} e_i - \frac{1}{2} (d_i - 1) \sum_{j \in N_i} \tilde{T}_{ik} + \sum_{j \in N_i} c_{ij} \]

\[ \Rightarrow \]

\[ \sum_{j \in N_i} \tilde{T}_{ij} = \frac{d_i}{d_i + 1} e_i + \frac{2}{d_i + 1} \sum_{j \in N_i} c_{ij} \]

\[ \Rightarrow \]

\[ \tilde{T}_{ij} = \frac{1}{2} e_i - \frac{1}{2} \sum_{k \in N_i \setminus \{j\}} \tilde{T}_{ik} + c_{ij} = \frac{1}{2} e_i - \frac{1}{2} \sum_{k \in N_i} \tilde{T}_{ik} + \frac{1}{2} \tilde{T}_{ij} + c_{ij} \]

\[ \Rightarrow \]

\[ \frac{1}{2} \tilde{T}_{ij} = \frac{1}{2} \left( e_i - \frac{d_i}{d_i + 1} e_i - \frac{2}{d_i + 1} \sum_{k \in N_i} c_{ik} \right) + c_{ij} \]

\[ \Rightarrow \]

\[ \tilde{T}_{ij} = \frac{1}{d_i + 1} e_i - \frac{1}{d_i + 1} \sum_{k \in N_i} c_{ik} + c_{ij} \]

Hence, by (20), we have

\[ t_{ij} = \frac{1}{2} e_i - \frac{1}{2} e_j - \frac{1}{2} \sum_{k \in N_i \setminus \{j\}} \left( \frac{1}{d_i + 1} e_i - \frac{1}{d_i + 1} \sum_{k' \in N_i} c_{ik'} + c_{ik} \right) \]

\[ + \frac{1}{2} \sum_{h \in N_j \setminus \{i\}} \left( \frac{1}{d_j + 1} e_j - \frac{1}{d_j + 1} \sum_{h' \in N_j} c_{jh'} + c_{jh} \right) + \frac{1}{2r} \ln \alpha_{ij} \]

\[ = \frac{1}{2} \left( 1 - \frac{d_i - 1}{d_i + 1} \right) e_i - \frac{1}{2} \left( 1 - \frac{d_i - 1}{d_i + 1} \right) e_j + C_{ij} \]

\[ = \frac{1}{d_i + 1} e_i - \frac{1}{d_i + 1} e_j + C_{ij}. \]