Supplementary Appendix to "A delegation-based theory of expertise"

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This supplement provides an analysis of max equilibria, in which the principal always chooses the higher offer. A max equilibrium exists if the bonus is not too large, and when it exists, it is unique. We characterize strategies, payoffs, and the bias and variance of the chosen offer in max equilibrium. We then characterize the max equilibrium for the variation of the model in which the unselected expert is indifferent over the implemented action.

S.1 Max Equilibrium

Here we provide a characterization of *max* equilibria — equilibria in which the experts play constant markup strategies and the principal always chooses the *highest* offer. The max equilibrium exists if and only if the bonus is sufficiently low; the threshold B_M can be negative, in which case no max equilibrium exists. When it does exist, it is unique. Intuitively, in the max equilibrium, the principal's strategy of choosing the higher offer implies that, conditional on being chosen, an expert must revise his belief and his markup downward. This downward force must be sufficiently large to ensure that the sum of markups is negative, so that the principal's choice of the higher offer is a best response. Hence, noise must be sufficiently large relative to the bonus for the max equilibrium to exist.

Recall the definition of z^* from (4) in the main paper.

Theorem S.1. *There exists a threshold B_M such that a max equilibrium exists if and only if $B \leq B_M$. When it exists, it is unique and characterized by $k_1^M = b_1 - \rho w(z^*)$ and $k_2^M = b_2 - \rho v(z^*)$, with $k_1^M - k_2^M = z^* \geq b_1 - b_2$ and $B_M \in [2\sigma^2 - \sqrt{\pi}\sigma(b_1 + b_2), 2\sigma^2 - 2\sqrt{\pi}\sigma \max(0, b_2)]$. For $B \leq B_M$,*

- $\bar{b}(k_1^M, k_2^M, H) = b_M = b_1 F(z^*) + b_2(1 - F(z^*)) + Bf(z^*)$;
- $Var(k_1^M, k_2^M, H) = \sigma^2 - 4\sigma^4 f^2(z^*) - 2\sigma^2 z^* f(z^*)(2F(z^*) - 1) + (z^*)^2 F(z^*)(1 - F(z^*))$.

The following corollaries are immediate from Theorems A.1 and S.1.

Corollary S.1. *A max equilibrium exists only if a min equilibrium exists; that is,* $B_M \leq B_m$.

Corollary S.2. *For* $B \leq B_M$, $k_1^m - b_1 = b_2 - k_2^M$ *and* $k_2^m - b_2 = b_1 - k_1^M$.

Before proving Theorem S.1, we state a lemma which is analogous to Lemma A.1.

Lemma S.1. *If both experts follow constant markup strategies* $a_j(s_j) = s_j + k_j$ *and the principal always chooses the higher offer, then*

$$\begin{aligned}
\bar{b}(k_1, k_2, H) &= k_1 + k_2 - \bar{b}(k_1, k_2, L); \\
Var(k_1, k_2, H) &= Var(k_1, k_2, L); \\
V(k_1, k_2, H) &= V(-k_1, -k_2, L); \\
U_i(k_1, k_2, H) &= -\sigma^2 - 2\sigma^2(k_i + k_j - 2b_i)f(z) - (k_i - b_i)^2 F(k_i - k_j) - (k_j - b_i)^2 F(k_j - k_i) \\
&\quad + BF(k_i - k_j).
\end{aligned}$$

Proof of Lemma S.1. Since the principal chooses the highest offer, she chooses a_i iff $s_j < s_i + k_i - k_j$. Using arguments similar to used in Lemma A.1, we find the expected utility of expert i:

$$\begin{aligned}
U_i(k_1, k_2, H) &= \int_{k_i - k_j}^{\infty} \mathbb{E}[-(a_j - \theta - b_i)^2 | s_j] f(s_j)\, ds_j + \int_{-\infty}^{k_i - k_j} \mathbb{E}[B - (a_i - \theta - b_i)^2 | s_j] f(s_j)\, ds_j \\
&= \int_{k_i - k_j}^{\infty} \left[-\left(k_j - b_i + \frac{s_j}{2} \right)^2 - \frac{\sigma^2}{2} \right] f(s_j)\, ds_j + \int_{-\infty}^{k_i - k_j} \left[B - \left(k_i - b_i - \frac{s_j}{2} \right)^2 - \frac{\sigma^2}{2} \right] f(s_j)\, ds_j \\
&= (B - (k_i - b_i)^2) F(k_i - k_j) - \sigma^2 - (k_j - b_i)^2 [1 - F(k_i - k_j)] - 2\sigma^2 (k_i + k_j - 2b_i) f(k_i - k_j).
\end{aligned}$$

In state θ the principal's action a is distributed as $\theta + \eta$, where $\eta \sim \max(\epsilon_1 + k_1, \epsilon_2 + k_2)$; $\epsilon_1, \epsilon_2 \sim N(0, \sigma^2)$, ϵ_1 and ϵ_2 are independent.

From Lemma A.2 the expected bias of the accepted offer is

$$\bar{b}(k_1, k_2, H) = \mathbb{E}\eta(k_1, k_2) = 2\sigma^2 f(k_1 - k_2) + k_2(1 - F(k_1 - k_2)) + k_1 F(k_1 - k_2).$$

The expected utility of the principal is

$$\begin{aligned}
V(k_1, k_2, H) &= -\mathbb{E}(a - \theta)^2 = -\mathbb{E}\eta^2(k_1, k_2) \\
&= -\sigma^2 - 2(k_1 + k_2)\sigma^2 f(k_1 - k_2) - k_2^2 - (k_1^2 - k_2^2) F(k_1 - k_2).
\end{aligned}$$

The variance of the chosen offer is

$$
\begin{aligned}
Var(k_1,k_2,H) =& - V(k_1,k_2,H) - \bar{b}^2(k_1,k_2,H) \\
=& \sigma^2 - 4\sigma^4 f^2(z) - 2\sigma^2 z f(z)(2F(z)-1) + z^2 F(z)(1-F(z)).
\end{aligned}
$$

□

Proof of Theorem S.1. The proof is analogous to that of Theorem A.1. The FOCs for experts are now:

$$
k_1 - b_1 + \rho \frac{f(k_1-k_2)}{F(k_1-k_2)} = 0 \tag{S.1}
$$

$$
k_2 - b_2 + \rho \frac{f(k_1-k_2)}{1-F(k_1-k_2)} = 0. \tag{S.2}
$$

Subtracting equation (S.1) from equation (S.2) yields (7). Principal optimality holds if and only if $k_1 + k_2 \leq 0$, or equivalently

$$
b_1 + b_2 - \rho \left[\frac{f(z^*)}{1-F(z^*)} + \frac{f(z^*)}{F(z^*)} \right] \leq 0, \tag{S.3}
$$

Define a function $n(B) = b_1 + b_2 - \rho[v(z(B)) + w(z(B)]$, where $z(B)$ is given by equation (4). For $B > 2\sigma^2$, we have $n(B) > 0$, and thus a max equilibrium does not exist. Observe further that $n(2\sigma^2) = b_1 + b_2 \geq 0$. Since $m(B) + n(B) = 2(b_1 + b_2)$ and $m'(B) < 0$, we have $n'(B) > 0$. It follows that if $n(0) \leq 0$, then there exists $B_M \in [0, 2\sigma^2]$ such that $n(B) \leq 0$ iff $B \leq B_M$. Therefore

$$
\left(\frac{B_M}{2} - \sigma^2 \right) [v(z(B_M)) + w(z(B_M))] = -(b_1 + b_2). \tag{S.4}
$$

Also $z(B_M)$ satisfies equation (4), and therefore

$$
\left(\frac{B_M}{2} - \sigma^2 \right) [v(z(B_M)) - w(z(B_M))] + z(B_M) = b_1 - b_2. \tag{S.5}
$$

From the previous discussion and (S.4) we have $B_M \leq 2\sigma^2$. Also (S.4) and the inequality $v(x) + w(x) \geq 2v(0) = \frac{2}{\sqrt{\pi}\sigma}$ give the lower bound

$$
\left(\frac{B_M}{2} - \sigma^2 \right) \frac{2}{\sqrt{\pi}\sigma} \geq -(b_1 + b_2).
$$

Summing (S.5) and (S.4), we get the upper bound

$$-2b_2 = \left(\frac{B_M}{2} - \sigma^2\right) v(z(B_M)) + z(B_M) \geq (B_M - 2\sigma^2)2v(0).$$

Finally, we compute the following:

$$\begin{aligned}
\bar{b}(k_1^M, k_2^M, H) &= 2\sigma^2 f(z^*) + k_1^M F(z^*) + k_2^M(1 - F(z^*)) \\
&= 2\sigma^2 f(z^*) + b_1 F(z^*) - \rho f(z^*) + b_2(1 - F(z^*)) - \rho f(z^*) \\
&= b_1 F(z^*) + b_2(1 - F(z^*)) + B f(z^*); \\
Var(k_1^M, k_2^M, H) &= \sigma^2 - 4\sigma^4 f^2(z^*) - 2\sigma^2 z^* f(z^*)(2F(z^*) - 1) + (z^*)^2 F(z^*)(1 - F(z^*)).
\end{aligned}$$

□

Recall the definition of $\Delta(z^*)$ from Corollary A.1.

Corollary S.3. *In the max equilibrium,*

$$\begin{aligned}
V(k_1^M, k_2^M, H) &= -\sigma^2 - b_1^2 F(z^*) - b_2^2(1 - F(z^*)) - B(b_1 + b_2) f(z^*) + \Delta(z^*) \\
U_1(k_1^M, k_2^M, H) &= -\sigma^2 - (b_1 - b_2)^2(1 - F(z^*)) + BF(z^*) + B(b_1 - b_2) f(z^*) + \Delta(z^*) \\
U_2(k_1^M, k_2^M, H) &= -\sigma^2 - (b_1 - b_2)^2 F(z^*) + B(1 - F(z^*)) - B(b_1 - b_2) f(z^*) + \Delta(z^*).
\end{aligned}$$

Proof. Lemma S.1 applied to Theorem S.1. □

S.2 Unselected Expert Indifferent over Actions

If the unselected expert is indifferent over the implemented action, the experts' expected payoffs in max equilibrium can be calculated as:

$$\begin{aligned}
U_i(k_i, k_j, H) &= \int_{-\infty}^{k_i - k_j} \left[B - \left(k_i - b - \frac{t}{2}\right)^2 - \frac{\sigma^2}{2}\right] f(t)\, dt \\
&= \left[B - \sigma^2 - (k_i - b)^2\right] F(k_i - k_j) - \left[2\sigma^2(k_i - b) - \frac{1}{2}\sigma^2(k_i - k_j)\right] f(k_i - k_j)
\end{aligned}$$

As in the min equilibrium, 0 is a lower bound for experts' equilibrium payoffs.

For the max equilibrium, the difference relative to the baseline model is the mirror image of the difference described earlier for the min equilibrium. Again, the bonus is reduced by quadratic losses, but in the max equilibrium this causes markups to decrease, as experts compete less aggressively to make the higher offer.

Proposition S.1. *Consider $b_1 = b_2 = b > 0$. If $\frac{b}{\sigma} > \frac{3\sqrt{\pi}}{4(\pi-1)}$, then no symmetric max equilibrium exists. If $\frac{b}{\sigma} \leq \frac{3\sqrt{\pi}}{4(\pi-1)}$, then a symmetric max equilibrium $k_1^M = k_2^M = k_M$ exists if and only if $B \in [B_1, B_2]$.*

Proof. Start with calculation of marginal utilities:

$$U_i'(k_i) = -2(k_i - b_i)F(k_i - k_j) - \left[\frac{\sigma^2}{2} + 2\rho + \frac{1}{4}(k_i + k_j - 2b)^2\right] f(k_i - k_j)$$

Consider the symmetric case: $k_1 = k_2 = k$. The FOCs give two critical points: $k = b - \sqrt{\pi}\sigma - \sqrt{\left(\pi - \frac{5}{2}\right)\sigma^2 + B}$ and $k = b - \sqrt{\pi}\sigma + \sqrt{\left(\pi - \frac{5}{2}\right)\sigma^2 + B}$.

Next, calculate second derivatives:

$$U_i''(k_i) = -2F(k_i - k_j)$$
$$- \left[2(k_i - b_i) + \frac{1}{2}(k_i + k_j - 2b_i) + \left(\frac{B}{2\sigma^2} - \frac{5}{4}\right)(k_i - k_j) - \frac{1}{8\sigma^2}(k_i - k_j)(k_i + k_j - 2b_i)^2\right] f(k_i - k_j).$$

We get that only $k^* = b - \sqrt{\pi}\sigma + \sqrt{\left(\pi - \frac{5}{2}\right)\sigma^2 + B}$ satisfies SOCs.

In order to satisfy principal optimality we need $k^* \leq 0$ or, equivalently, both $\frac{b}{\sigma} \leq \sqrt{\pi}$ and $B \leq b^2 - \sqrt{\pi}b\sigma + \frac{5}{2}\sigma^2$.

Calculating, we get that $U_i(k^*, k^*, H) = \frac{(\pi-1)\sigma}{\sqrt{\pi}}\sqrt{\left(\pi - \frac{5}{2}\right)\sigma^2 + B} - (\pi - \frac{7}{4})\sigma^2$ (the same as in the min equilibrium). As in the min equilibrium case, a necessary condition is $B \geq \left(\frac{5}{2} - \frac{3\pi(8\pi-11)}{16(\pi-1)^2}\right)\sigma^2$.

From previous arguments, a necessary for max equilibrium to exist is $B \in \left[\left(\frac{5}{2} - \frac{3\pi(8\pi-11)}{16(\pi-1)^2}\right)\sigma^2, b^2 - 2\sqrt{\pi}b\sigma + \frac{5}{2}\sigma^2\right)$ and $\frac{b}{\sigma} \leq \sqrt{\pi}$. This interval is non-empty if and only if $\frac{b}{\sigma} \leq \frac{3\sqrt{\pi}}{4(\pi-1)}$. Note also that $B \leq b^2 - 2\sqrt{\pi}b\sigma + \frac{5}{2}\sigma^2 \leq \frac{5}{2}\sigma^2$.

To finish the proof we show that if B lies on this interval, then $k = k^*$ is not only a local, but also a global maximum of $U_1(k, k^*, H)$.

Denote $r(k) = -2(k - b) - \left[\frac{\sigma^2}{2} + 2\rho + \frac{1}{4}(k + k^* - 2b)^2\right] w(k - k^*)$ (recall that $w(k - k^*) = \frac{f(k-k^*)}{F(k-k^*)}$).

Then $U_1'(k) = -2(k - b)F(k - k^*) - \left[\frac{\sigma^2}{2} + 2\rho + \frac{1}{4}(k + k^* - 2b)^2\right] f(k - k^*) = F(k - k^*)r(k)$ and $sign(U_1'(k)) = sign(r(k))$.

First and second derivatives of $r(k)$ are:

$r'(k) = -2 - \left[\frac{\sigma^2}{2} + 2\rho + \frac{1}{4}(k + k^* - 2b)^2\right] w'(k - k^*) - \frac{1}{2}(k + k^* - 2b)w(k - k^*)$;

$r''(k) = -\left[\frac{\sigma^2}{2} + 2\rho + \frac{1}{4}(k + k^* - 2b)^2\right] w''(k - k^*) - (k + k^* - 2b)w'(k - k^*) - \frac{1}{2}w(k - k^*)$.

Notice that as $\frac{b}{\sigma} \leq \sqrt{\pi}$, $B \leq b^2 - \sqrt{\pi} b\sigma + \frac{5}{2}\sigma^2 \leq \frac{5}{2}\sigma^2$ and $k^* = b - \sqrt{\pi}\sigma + \sqrt{\left(\pi - \frac{5}{2}\right)\sigma^2 + B} \leq b$.

a) On interval $k > b$ $U'(k) < 0$, so there is no candidate for maximum there.

b) On interval $k < b$ we also have $k + k^* - 2b \leq 0$. As $w > 0$, $w' < 0$, $w'' > 0$, we have $r''(k) < 0$.

As $r'(-\infty) > 0$ and $r'(k^*) < 0$, there exists $k^{**} < k^* < b$: $r(k)$ is increasing for $k < k^{**}$, $r(k)$ is decreasing for $k > k^{**}$. As also $r(-\infty) < 0$, $r(k^* - 0) > 0$ and $r(k^* + 0) < 0$, there exists k_0: $r(k) > 0$ only on (k_0, k^*). Therefore, $U_1(k)$ is decreasing on $k < k_0$, increasing on (k_0, k^*), decreasing on (k^*, b). Hence, k^* is a global maximum if $U_1(k^*) \geq U_1(-\infty) = 0$, which has already been shown. $\square$