Final Exam

Directions: You have two hours to complete the exam. There is a total of seven (7) questions. The maximum number of points for each of the questions are given in brackets. The whole exam is worth 100 points. Feel free to consult your notes and any books that you may find useful. Write clearly. Good Luck!

Name:______________________________
1. Let $y_t$, $c_t$, and $i_t$ denote the time series of logarithmic national income, consumption, and investment, respectively, each of which may be described by an I(1) process. Briefly discuss the problems associated with using a standard VAR approach for jointly modeling the growth rates in the three variables. How would you formulate a (sensible) multivariate time series model to allow for a more meaningful empirical analysis of the joint dynamic dependencies?
2. Consider the AR(2) stochastic process,

\[ y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + u_t, \]

where \( u_t \) is white noise. Rewrite the process with \( \Delta y_t = (1-L)y_t \) on the left-hand-side, and \( y_{t-1} \) and \( \Delta y_{t-1} \) on the right-hand-side. Characterize and discuss the limiting distribution of the "t-statistic" associated with the estimated regression coefficient for \( y_{t-1} \). How does the distribution depend upon the values of \( \phi_1 \) and \( \phi_2 \)?
3. Consider the MA(1) process for $y_t$:

$$y_t = \varepsilon_t - 1.5\varepsilon_{t-1},$$

where $\varepsilon_t$ is a white noise process with mean zero and unit variance.

a. Is this process for $\{y_t\}$ covariance stationary?

b. Derive the autocovariance generating function for $y_t$?

c. Sketch the spectrum for $y_t$?

d. Is the process for $\{y_t\}$ invertible?

e. Derive an invertible representation for $\{y_t\}$. Is this representation unique?

f. You need to make a forecast for $y_{T+1}$. The only piece of information that you have is the realization at time $T$, $y_T = 1.0$. What would be your forecast?

g. What is the variance of the forecast error associated with your forecast in part f?

h. Now suppose that you have data on $\{y_t\}_{t=1}^T$, where $T$ is very large. How would you construct a forecast for $y_{T+1}$ in this situation?

i. What would be the (approximate) variance of the forecast error associated with the forecast in part h? How does this value compare to the variance of the forecast in part f?
Consider the ARMA(1,1) model,

\[ y_t = \phi_1 y_{t-1} + \theta_1 \epsilon_{t-1} + \epsilon_t, \]

where \( E_{t-1}(\epsilon_t) = 0. \) Discuss how you might estimate this model using:

1. Approximate maximum likelihood.
2. Exact maximum likelihood.
3. GMM.

Be sure to state all of your assumptions clearly.
5. Consider the AR(1)-GARCH(1,1) model for $y_t$,

\[ y_t = \phi y_{t-1} + \epsilon_t \]
\[ \epsilon_t | I_{t-1} \sim N(0, \sigma^2_t) \]
\[ \sigma^2_t = \omega + \alpha \epsilon^2_{t-1} + \beta \sigma^2_{t-1} \]

where $|\phi|<1$, $\omega>0$, $\alpha>0$, $\beta>0$, $\alpha+\beta<1$, and $I_{t-1}$ denotes the information set available at time $t-1$.

a. Derive the optimal (in a mean square error sense) 1- and 2-steps ahead forecasts for $y_{t+1}$ and $y_{t+2}$ given $I_t$, along with their 95% confidence intervals.

b. Derive the limiting, as $h \to \infty$, optimal forecast for $y_{t+h}$ conditional on $I_t$ and the corresponding forecast error interval.
6. Consider the continuous time Random Walk model for the logarithmic price process $p(t)$,
\[ dp(t) = \mu \, dt + \sigma \, dW(t) , \quad 0 \leq t \leq S , \]
where $W(t)$ denotes a standard Brownian motion. Let,
\[ r_{\frac{t}{n}} = p(t) - p(t - (1/n)) , \quad \text{t = 1, 2, ..., T}, \]
denote the corresponding $T=n\cdot S$ discretely sampled $(1/n)$-period returns. It follows by standard arguments that
\[ r_{\frac{t}{n}} \text{ i.i.d. } N(\mu/n, \sigma^2/n). \]
a. Derive the maximum likelihood estimate (MLE) and associated asymptotic standard errors for $\mu$ and $\sigma^2$ based on the discretely sampled returns.

b. Are the MLEs for $\mu$ and $\sigma^2$ consistent as the number of discrete-time observations (T), the span of the data (S), and the sampling frequency (n) increases? Discuss and interpret your findings.
7. Suppose that $y_t$ follows a random walk,

$$y_t = y_{t-1} + u_t,$$

where $u_t$ is a white noise process. Consider the regression of $y_t$ on a deterministic time trend,

$$y_t = \beta t + \epsilon_t.$$

a. Show that the OLS estimate for $\beta$ converges in probability to zero.

b. Show that the $R^2$ from the regression does not converge to zero, but rather is $O_p(1)$.

Now consider the regression of $y_t$ on a constant and a deterministic time trend,

$$y_t = \alpha + \beta t + \epsilon_t.$$

c. What are the probability limits of the OLS estimates for $\alpha$ and $\beta$ from this regression?