A Retrospective View of Hicks' *Capital and Time: A Neo-Austrian Theory*

by

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1. Introduction

In 1973 Sir John Hicks published *Capital and Time: A Neo-Austrian Theory*. This was his third book with the word "capital" in its title, the first being his classic *Value and Capital* [1939] and the second being *Capital and Growth* [1965]. It departed significantly from his earlier work by assuming that the technology of an economy consisted of a set of neo-Austrian production processes in which a time sequence of inputs $\{a_t\}$ produces a time sequence of outputs $\{b_t\}$.

In June, 1974 I published a review article in the *Journal of Economic Literature* entitled "Synthesizing the Neo-Austrian and Alternative Approaches to Capital Theory: A Survey" using Hicks' book as a filter to select a list of topics for discussion.[2] Now, with almost 30 years of hindsight, I will revisit some of the problems that, in my view, remain both unsolved and important.

First, however, I point out that reading Hicks in 2002 is as much a delight as it was in 1973. His style is refreshingly old-fashioned, focused on answering economic questions with mathematics used only as a means of achieving those answers. For example, Hicks' view of the von Neumann model is that it is logically elegant, but that "… the categories with which it works are not very recognizable as economic categories;

so to make economic sense of its propositions translation is required. One has got so far away from the regular economic concepts that the translation is not at all an easy matter" [Hicks, 1973, p. 6].

But, as I shall argue below, this bias in favor of "regular economic concepts" comes at the cost of generality. As a result the primary contributions of *Capital and Time: A Neo-Austrian Theory* are pedagogical. Hicks' elegant examination of simple problems serves to deepen our economic understanding of many capital theory principles such as, for example, the duality between the factor-price frontier and the optimal transformation frontier. Yet one is left with a sense of unease that treacherous territory may lie just beyond these simple examples.

2. The Hicks Neo-Austrian Technology and Truncation

The Hicks technology is based on the Austrian tradition of Böhm-Bawerk, Wicksell, and Hayek in which a flow of inputs over time produces output at a later point in time. Hicks generalizes this idea so that a *production process* consists of a time sequence of inputs $\{a_t\}$ that produces an associated time sequence of outputs $\{b_t\}$. The Hicks neo-Austrian *technology* is the set of all such feasible production processes.

It is assumed that homogeneous labor is the only input and that there is only one type of homogeneous output, which Hicks identifies simply as "goods," although I prefer to interpret output as the quantity of a single type of consumption good [Hicks, 1973, p. 37]. Using this single consumption good as the numeraire, b_t measures both the physical quantity and the value of output. Then letting w_t denote the real wage rate (in terms of the consumption good) during period *t*, a production process yields a net output stream given by

$$\{q_t\}_{t=0}^{n} = \{(b_t - w_t a_t)\}_{t=0}^{n} \quad . \tag{2.1}$$

More than one production process may be used during any time period, and over time the economy may or may not converge to a steady-state equilibrium in which one most profitable production process is employed.

Hicks also assumes that $a_t > 0$ and $b_t = 0$ for $t = 0,1,\dots,m-1$. He then defines the *construction period* as the m time periods $t = 0,1,\dots,m-1$ during which labor inputs are employed, but there is no output of the consumption good [Hicks, 1973, p. 15]. It suffices here to point out the obvious fact that this technology is extraordinarily simple; see Burmeister [1974] for some details. However, it does enable Hicks to focus on some of the economic questions that arise from the pure role of time, without, for example, having to deal with the complications of heterogeneous inputs and outputs.

There is often some ambiguity in discrete-time models because the end of one time period coincides with the beginning of the next. To avoid inconsistencies in the Hicks neo-Austrian model, one can interpret labor input for a production process as the number of workers employed at the beginning of period t, who then work for the whole period and are paid a real wage rate at the end of period t. The output of the consumption good, on the other hand, is realized only at the end of period t. The reasons for these timing conventions are explained elsewhere [Burmeister, 1974, pp. 417-418] and need not concern us here.

Now assume that the real-wage rate is constant, $w_t = w$, and also that there is a constant per period (Hicks uses weeks) real rate of interest (in terms of the consumption good) denoted by r. Then *the capital value of the process* at the beginning of period 0 as

(2.2) $$k_0 = \sum_{t=0}^{n} q_t R^{-(t+1)} = \sum_{t=0}^{n} \left(b_t - wa_t\right) R^{-(t+1)}$$

where the interest rate factor is $R \equiv 1 + r$. Note that k_0 is simply the present discounted value of the production process. Hicks assumes that capital markets are in equilibrium so that $k_0 = 0$ [Hicks, 1973, p. 32].[3] Note that $k_0 = 0$ is equivalent to a zero-profit condition when all inputs and outputs are measured in terms of discounted prices.

More generally, the capital value of the process at the beginning of any time period t is

(2.3) $$k_t = \sum_{i=t}^{n} q_i R^{-(i+1-t)} = \sum_{i=t}^{n} \left(b_i - wa_i\right) R^{-(i+1-t)}$$

so that

(2.4) $$k_t = \left(q_t + k_{t+1}\right) R^{-1} \ .$$

Thus the capital value of the production process at the beginning of period t is equal to the value of net output at the end of period t, denoted by q_t, plus the capital value at the beginning of period $t+1$, both discounted.[4]

Given the real wage rate w and the interest rate r, the economic lifetime of a project, Ω, is determined by maximizing (2.2) over the terminal time n under the very strong assumption that the production process can by *truncated* at any time. That is, a feasible production process $\left\{\left(a_t, b_t\right)\right\}_{t=0}^{n}$ can be *truncated* if, and only if, the shorter process

(2.5) $$\left\{\left(a'_t, b'_t\right)\right\}_{t=0}^{n} = \left\{\left(a_0, b_0\right), \left(a_1, b_1\right), \ldots, \left(a_m, b_m\right), \left(0,0\right), \ldots, \left(0,0\right)\right\}$$

is also technologically feasible for all $m \le n$. It is important to note that the conventional free disposal assumption does not imply the truncation property (2.5). Free disposal implies that the same or less output can be produced with additional inputs, but it implies nothing about what can be produced with fewer inputs.

Denote the present discounted value of the project at the beginning of period 0, when it is operated through to period *T*, as

(2.6) $$k\left(0, r, T; w\right) \equiv \sum_{t=0}^{T} q_t R^{-(t+1)} = \sum_{t=0}^{T} \left(b_t - w a_t\right) R^{-(t+1)} \quad \text{for} \quad T = 0, 1, \ldots, n .$$

Hicks denotes this value by the ambiguous notation k_0 , as we shall do when no confusion is possible.[5] We assume that the process is viable at the given real wage rate and interest rate so that this present discounted value is strictly positive for some value of *T*.

The optimal lifetime or duration of a production process is defined as a value of Ω for which

(2.7) $$k\left(0, r, \Omega; w\right) > k\left(0, r, T; w\right) \quad \text{for all} \quad T \ne \Omega .$$

Using the Hicks notation, this definition rules out ties and implies that

(2.8) $$k_t > 0 \quad \text{for all} \quad t = 0, 1, \ldots, \Omega \;;$$

see [Hicks, 1973, p. 18 and especially footnote 2].

3. Hicks' Fundamental Theorem

Given all of the stated assumptions, Hicks proves what he calls a *Fundamental Theorem*, namely given a real wage rate, a rise (fall) in the rate of interest will lower (raise) the capital value of the production process for all time horizons $0 \le t \le \Omega$ [Hicks, 1973, p. 19]. The following numerical example proves that this result does not hold without the truncation property. Consider a production process with net outputs $\left\{q_0 = -1, q_1 = 2.3, q_2 = -1.32\right\}$ at some given real wage rate. The capital values $k_t\left(r\right)$ computed from (1.3) for interest rates $r = \left\{0, 0.1, 0.15, 0.2, 1\right\}$ are given in Table 1.

$r =$	$k_0(r)$	$k_1(r)$	$k_2(r)$
0	– 0.02	0.98	– 1.32
0.10	0	1	– 1.2
0.15	0.00164379	1.001890	– 1.147826
0.20	0	1	– 1.1
1.0	– 0.09	0.82	– 0.66

Table 1
Process not truncated.

Note that as the rate of interest is increased, the capital value at the beginning of period 1 does not always fall, but rather it first rises and then falls. Because $q_2 < 0$, it is more profitable to truncate the process at the beginning of period 2 no matter what the interest rate. This fact is also reflected by the negative values of $k_2(r)$. Note also that both $k_0(0.1) = 0$ and $k_0(0.2) = 0$ so that this untruncated process has *two* internal rates of return. It will be operated only for interest rates satisfying $0.1 \le r \le 0.2$.

When truncation is allowed, the optimal lifetime is $\Omega = 1$. Table 2 shows the corresponding capital values. Note that now both the capital value at the beginning of period 0 and the capital value at the beginning of period 1 fall as the interest rate increases, as asserted by Hicks' Fundamental Theorem.

$r =$	$k_0(r)$	$k_1(r)$
0	1.3	2.3
0.10	0.991736	2.090909
0.15	0.869565	2
0.20	0.763889	1.916667
1.0	0.075	1.15

Table 2
Process optimally truncated at $\Omega = 2$.

It follows immediately from the Fundamental Theorem that if there exists a positive rate of interest $r = r_1$ for which $k_0 = 0$, then this value r_1 is unique. In addition, such an r_1 always exists if the production process is viable at the prevailing real wage rate, and it is equal to the internal rate of return for the production process [Hicks, 1973, pp. 18-21]. Although Hicks' exposition of his Fundamental Theorem is both elegant and based on economic principles, the result had been anticipated by others who derived sufficient conditions for the uniqueness of the internal rate of return in much more general cases than the one considered by Hicks; see, for example, [Arrow and Levhari, 1969], [Flemming and Wright, 1971], and especially [Sen, 1975].

This Fundamental Theorem is illustrated in Figure 1. For this numerical example the production process has net outputs $\{q_t\}_{t=0}^{4} = \{-1, 0.6, 0.75, -0.9, 1\}$. At interest rates $r = 0$ and $r = 0.1$ the optimal lifetime of the process is $\Omega = 4$, while at interest rates $r = 0.15$ and $r = 0.216515$ it is $\Omega = 2$. As the interest rises from 0, the capital value curves fall. Therefore there exists exactly one interest rate, $r = 0.216515$, for which $k_0 = 0$. This interest rate is the unique internal rate of return for the optimally truncated process.

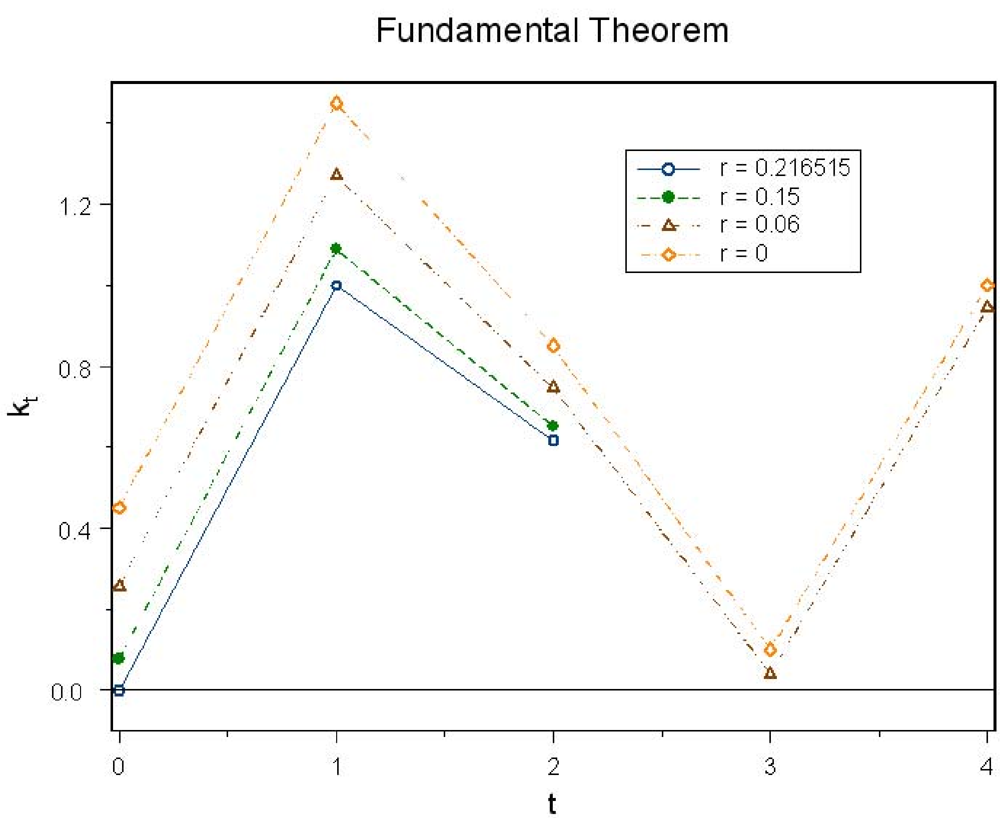

Figure 1

For completeness in Appendix A we sketch a proof of Hicks' Fundamental Theorem using our timing convention for inputs and outputs.

4. The "Cambridge Capital Theory Controversies"

No historical account of capital theory is complete without mention of the so-called "Cambridge Capital Theory Controversies" that peaked in the mid 1960's. We need to first briefly remind ourselves of a few key features of this controversy before we can see how Hicks' *Capital and Time* fits into the picture.

The issue concerns non-joint production, constant-returns-to-scale technologies with one primary factor, labor. Many people once believed that many of the steady state results obtained for a one-capital good Solow/Swan model also held for more complex technologies with many different types of capital goods. In particular, by analogy with the familiar one-capital good models, some thought—incorrectly as it turns out—that steady-state per capita consumption always increases with decreases in the steady-state rate of interest, so long as the rate of interest remains above the golden rule value $r = g$ (where g is the exogenous growth rate of labor). The mistaken intuition for such thoughts was based on the notion of "capital deepening" whereby an economy in a steady state with a low rate of interest (but still bigger than g) would have more "capital" than at a higher rate of interest, and hence it would be able to produce more per capita consumption. The latter statement is, of course, correct for models with only one type of capital good.

Numerical examples originating from the Cambridge, England, school of thought immediately convinced those in Cambridge, Mass. that such "capital deepening" results do not necessarily hold in a world with more than one capital good. Figure 2 shows the factor-price frontiers for production processes A and B. These frontiers trace the steady-state relationship between the rate of interest and the real wage rate. For any given interest rate, competition insures that the corresponding equilibrium wage rate is on the highest frontier using the corresponding production process. Alternatively, as Hicks prefers, one can take w as given, and then the corresponding point on the factor-price frontier gives the equilibrium value of r.

The processes A and B have *switch points* at interest rates r_1 and r_2. Therefore process A is used for all interest rates satisfying either $0 \leq r \leq r_1$ or $r_2 \leq r \leq r_{\max}$, where $r_{\max}$ is the largest steady-state interest rate for which process A is viable. Similarly, process B is used for all interest rates satisfying $r_1 \leq r \leq r_2$, and at the two switch points, both process can coexist. Thus Figure 2 illustrates what is called the *reswitching of techniques*, though here I use the term "process" instead of "technique" to be consistent with Hicks.

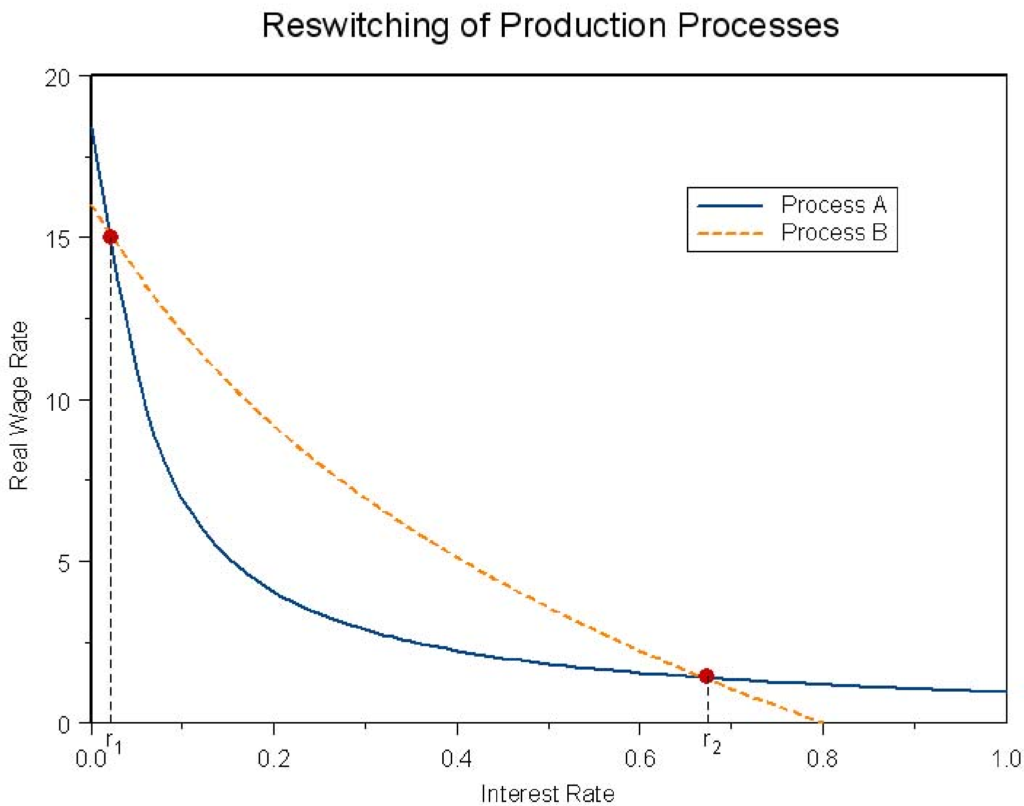

Figure 2

Denoting the factor-price frontiers illustrated in Figure 2 by $w = f_A(r)$ and $w = f_B(r)$, and assuming that labor does not grow, it is easily shown that per capita consumption is equal to $c = f_A(0)$ when process A is employed and is equal to $c = f_B(0)$ when process B is employed. Therefore, the so-called *paradoxical steady-state consumption behavior* illustrated in Figure 3 exists. That is, as the steady-state interest rate decreases from, say, 0.8 to 0.2, the economy switches from using production process A to production process B and steady-state consumption *falls*. This clearly demonstrates that no notion of "capital deepening" can be valid in models with more than one capital good.

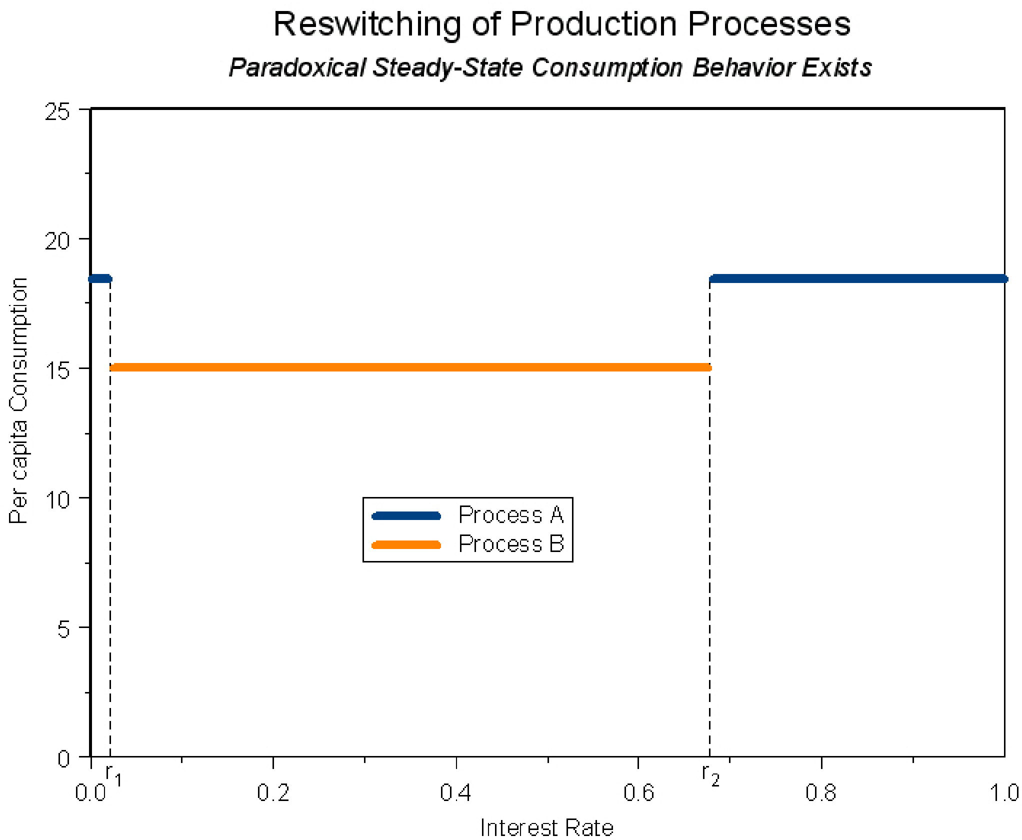

Figure 3

Moreover—and there has been much confusion about this point—while the existence of reswitching clearly reveals the existence of such paradoxical consumption behavior, this behavior can arise even in models for which there is no reswitching. This is illustrated in Figure 4. Here process C is employed at high interest rates, process B is employed at intermediate interest rates, and process A is employed at low interest rates. Yet steady-state consumption decreases as the interest rate is lowered from, say, 0.8 to 0.2.

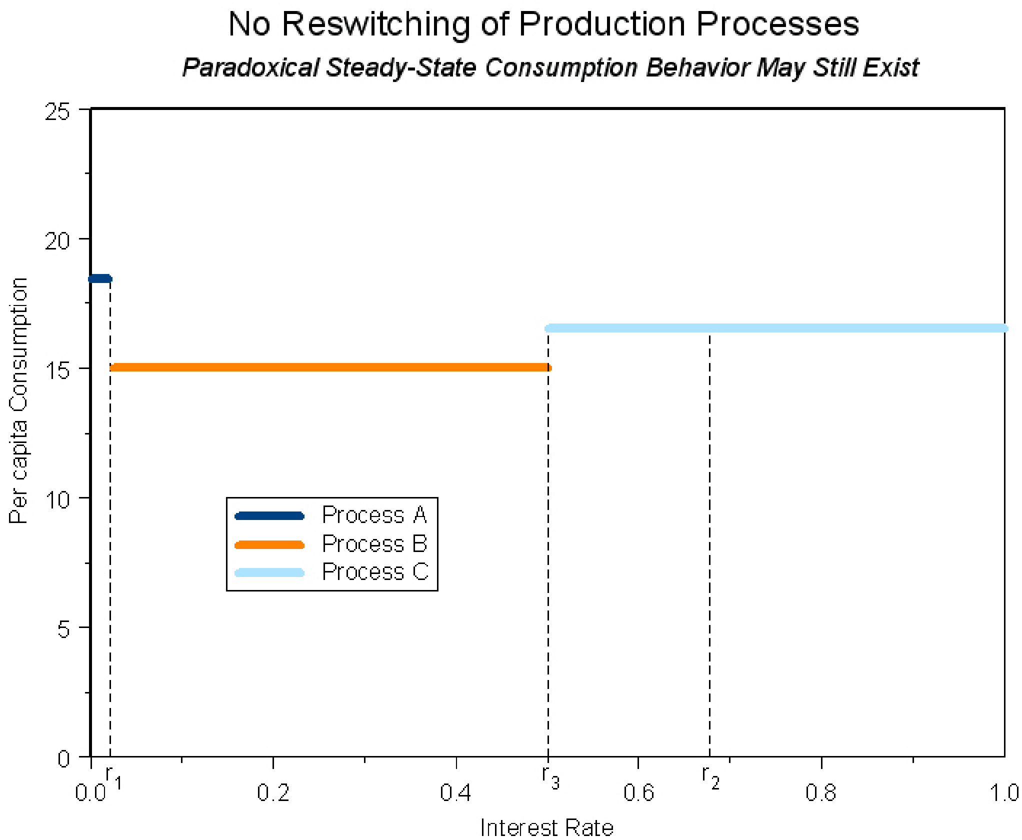

Figure 4

In retrospect it is clear that steady-state comparisons are not of very much economic interest because they do not represent choices among feasible alternatives. Given that an economy is in a steady-state equilibrium at some interest rate, the relevant economic questions concern the properties of the feasible dynamic paths starting from this steady state as an initial condition. And there is nothing strange or paradoxical about these feasible paths. Neoclassical economics is alive and well, even when there are many different types of capital goods.

5. The Factor-Price Frontier and Duality for the Hicks Neo-Austrian Model

We now take as given any real wage rate w_i for which a production process is viable, and let r_i denote the corresponding unique internal rate of return for which the capital value at the beginning of period 0 is equal to 0, the condition for equilibrium in Hicks' capital market. The set of such equilibrium points $\{w_i, r_i\}$ defines a function

(5.1) $$r = \phi(w)$$

which Hicks calls the *efficiency curve* of the process. He objects to the more common terminology *factor-price curve* on the grounds that the interest rate is not the price of a factor.[6] Nevertheless, there is considerable advantage to using terminology consistent with the main body of economic literature, and here we shall refer to the relationship defined by (5.1) as the *factor-price curve*.

More precisely, since it is easily shown that $\phi'(w) < 0$; hence $\phi(w)$ may be inverted to write our *factor-price curve* in the conventional form

(5.2) $$w = f(r) \quad \text{with} \quad f'(r) < 0 \ .$$[7]

Such a factor-price curve exists for every production process. The outer envelope of the set of these factor-price curves defines the *factor-price frontier* for the economy, which is denoted by

(5.3) $$w = F(r) \quad \text{with} \quad F'(r) < 0 \ .$$

For any given feasible interest rate *r*, an economy in competitive equilibrium always acts to maximize the real wage rate *w* by operating on this factor-price frontier.

Along a factor-price curve and the factor-price frontier, duration is always at its optimal length. Hicks also defines a *restricted efficiency curve* along which the optimal duration is fixed at the value appropriate for some interest rate [Hicks, 1973, pp. 66-67]. Our corresponding *restricted factor-price curve* is denoted by

(5.4) $$w = \psi(r) \ .$$

For a given production process, the optimal duration may change with the interest rate. For each optimal duration, there is a different $w = \psi(r)$ function, and their outer envelope is the factor-price curve given by (5.2).

In addition, using the capital market equilibrium condition $k_0 = 0$ we may compute (5.4) as

(5.5) $$w = \psi(r) = \frac{\sum_{t=0}^{\Omega_1} b_t (1+r)^{-(t+1)}}{\sum_{t=0}^{\Omega_1} a_t (1+r)^{-(t+1)}} = \frac{\sum_{t=0}^{\Omega_1} b_t (1+r)^{-t}}{\sum_{t=0}^{\Omega_1} a_t (1+r)^{-t}}$$

where Ω_1 is the optimal duration at the interest rate r_1.

We now need to examine the quantity side of the model before we can discuss the duality results. Hicks assumes that there are constant-returns-to-scale in the sense that production processes can be replicated. Let τ denote the beginning of the current time period ("week"), and let $x_{\tau-t}$ denote the number of unit production processes that began operation *t* periods ago. Let either $w = w_1$ or $r = r_1$ be given, where $w_1 = f(r_1)$, and let Ω_1 be the corresponding optimal duration. The total labor input and the current output (of the consumption good) are

(5.6) $$A_\tau = \sum_{t=0}^{\Omega_1} x_{\tau-t} a_t$$

and

(5.7) $$B_\tau = \sum_{t=0}^{\Omega_1} x_{\tau-t} b_t \ ,$$

respectively. In a steady-state equilibrium the number of starts must grow at the constant rate of *g* per period and hence

(5.8) $$x_\tau = x_0 (1+g)^\tau$$

where x_0 determines the scale of the system. Then (5.6) and (5.7) are replaced by

(5.9) $$A_\tau = \sum_{t=0}^{\Omega_1} x_0 (1+g)^{\tau-t} a_t = x_0 \left(\sum_{t=0}^{\Omega_1} a_t (1+g)^{-t} \right) (1+g)^\tau$$

and

(5.10) $$B_\tau = \sum_{t=0}^{\Omega_1} x_0 (1+g)^{\tau-t} b_t = x_0 \left(\sum_{t=0}^{\Omega_1} b_t (1+g)^{-t} \right) (1+g)^\tau \ ,$$

respectively. Similarly, in a steady-state equilibrium the current value of capital is

(5.11) $$K_\tau = \sum_{t=0}^{\Omega_1} x_{\tau-t} k_t = \sum_{t=0}^{\Omega_1} x_0 (1+g)^{\tau-t} k_t = x_0 \left(\sum_{t=0}^{\Omega_1} k_t (1+g)^{-t} \right) (1+g)^\tau \ .$$

From (5.9) and (5.10) we see that per capita consumption is

(5.12)
$$c \equiv \frac{C_\tau}{L_\tau} \equiv \frac{B_\tau}{A_\tau} = \frac{\sum_{t=0}^{\Omega_1} b_t (1+g)^{-t}}{\sum_{t=0}^{\Omega_1} a_t (1+g)^{-t}} = \psi(g) \ .$$

This is Hicks' duality result: Constant per capita consumption depends on g in exactly the same way that the real wage rate depends on r in (5.5). The Golden Rule theorem that $c = w$ at $r = g$ and other well-known results follow trivially.

The duality between steady-state per capita consumption and the real wage rate is fundamental, and the Hicks neo-Austrian model is only one example of it. The result was first proved by von Weizsäcker [1963] and subsequently was developed by Bruno [1969] and Hicks [1965] for Leontief technologies. Burmeister and Kuga [1970] proved the result for a neoclassical model with heterogeneous capital goods. The concept of a factor-price frontier originates with Samuelson [1957], and he provides an excellent review in [Samuelson, 1983, pp. 464-471].

6. Roundaboutness and Other Doomed Austrian Concepts

Consider three alternative production processes:

Process A = $\{(14,0),(14,0),(14,0),(14,0)\,(0,10)\}$

Process B = $\{(20,0),(20,0),(20,0)\,(0,10)\}$

Process C = $\{\,(35,0),(35,0)\,(0,10)\}$.

Then using (5.3) or (5.4) we compute that Process A is used for $0 \le r \le 0.137009$, Process B is used for $0.137009 \le r \le 0.318729$, and Process C is used for $r \ge 0.318729$.

One of the many possible definitions of the "average period of production" or "roundaboutness." is

$$\theta = \frac{\sum_{t=0}^{\Omega} (t+1) a_t}{\sum_{t=0}^{\Omega} a_t} \ ,$$

and for these three processes we have

$$\theta^A = 2.5$$
$$\theta^B = 2$$
$$\theta^C = 1.5 \quad .$$

This provides an example of:

Property 1:
The "degree of roundaboutness" increases as the interest rate falls.

Now taking the simple no-growth case with $g = 0$, we can compute from (5.12) that the steady-state per capita consumption levels for the three processes are:

$$c^A = 0.178571$$
$$c^B = 0.166667$$
$$c^C = 0.142857 \quad .$$

We thus have an example of:

Property 2:
Per capita consumption rises as the interest rate falls.

Finally, the current value of capital with $g = 0$ is computed from (5.11):

$$K^A = 21.646980$$
$$K^B = 15.659341$$
$$K^C = 10.119048 \quad .$$

This gives us:

Property 3:
Capital values rise as the interest rate falls.

These three Austrian properties hold only for special cases such as the one given here and are not generally valid. By 1973 when Hicks wrote *Capital and Time,* he was quite aware of this fact.[8] Indeed, the essence of Hicks' approach is to generalize the Austrian notion of output at a point in time to one in which output is a stream over time, and with this generalization the whole notion of "roundaboutness" collapses [Hicks, 1973, pp. 8-9].

In Chapter IV Hicks acknowledges that his production processes can exhibit reswitching, exactly as we have described it in Section 4 above. The existence of reswitching immediately provides a counterexample to Properties 1, 2, and 3 above. He calls reswitching "a *curiosum*," yet acknowledges that its importance is "… for the much more substantial issue that lies behind it." [Hicks, 1973, p. 41]. And that substantial issue is that the rate of interest cannot be used to characterize differences among production processes. Hicks does not seem to recognize that the failure of Properties 1, 2, and 3 can arise even in models for which no reswitching exists, as illustrated by our Figure 4; Bruno, Burmeister, and Sheshinski [1966] first made this observation and explained why. The Hicks solution is to assume that every production process has a *Simple Profile*, thereby circumventing all of these difficulties [Hicks, 1973, pp. 41-42].[9]

Some bad ideas have amazing endurance, and "roundaboutness" seems to be one of them. For example, Yeager's 1976 paper "Towards Understanding Some Paradoxes in Capital Theory" won a prize for the best *Economic Inquiry* publication in that year. Yet it is wrong, as Yeager graciously and candidly acknowledged in 1978; see [Yeager, 1976] and [Burmeister and Yeager, 1978].[10]

Earlier, before the possibility of reswitching was recognized, Samuelson fell into the similar error of asserting that economies with a lower steady-state interest rate had more "capital" and therefore were able to produce more per capita consumption, though with diminishing returns; see [Samuelson, 1964, Appendix to Chapter 28: Interest and Capital, pp. 594-600]. Samuelson has also candidly admitted his mistake; see especially [Samuelson, 1966].

Two more obscure Austrian-like properties deserve mention:

Property 4:
The economic lifetime of a machine increases with decreases in the interest rate.

Property 5:
The optimal durations for production processes in use increases with decreases in the interest rate.

Properties 4 and 5 also are not generally true. Hagemann and Kurz have constructed a clever example—but more complex than the ones we have considered here—that shows them both to be false; see [Hagemann and Kurz, 1976] and [Kurz and Salvadori, 1995, pp. 212-216].

7. The Neo-Austrian Approach as a Generalized von Newmann Model

I have summarized a generalized von Neumann model in Appendix B. Now I provide a simple example to illustrate how the Hicks neo-Austrian model is but a special case of this more general framework.

Consider the production process $\{(a_1, 0), (a_2, b_2), (a_3, 1)\}$. The generalized von Neumann interpretation is that Activity 1 uses a_1 workers employed for one period produce one new machine; Activity 2 uses a_2 workers together with one new machine employed for one period to produce jointly one one-year-old machine and b_2 units of the consumption good; and Activity 3 uses a_3 workers together with one one-year-old machine to produce one unit of the consumption good. In general, Activity $j+1$ is associated with the j^{th} period of production in Hicks' notation. Therefore the optimal duration problem is automatically solved once we identify which activities are operated at positive intensity levels in the von Neumann solution.

In the notation of Appendix B, we have

(7.1) $$A_0 = (a_1\ a_2\ a_3) ,$$

(7.2) $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} ,$$

and

(7.3) $$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b_2 & 1 \end{bmatrix} .$$

We assume that all three activities are operated at a positive intensity level for the given wage rate *w* and interest rate *r*. Then equation (B.10) in Appendix B implies

(7.4) $$wA_0 + (1+r)A = pB$$

or

(7.5) $$wA_0 = p\left[B-(1+r)A\right] .$$

This example is especially easy to solve because the number of activities is equal to the number of goods. Therefore for given *r* such that $\left[B-(1+r)A\right]^{-1}$ exists, (7.5) can be solved for

(7.6) $$p(r) = wA_0\left[B-(1+r)A\right]^{-1} .$$

The equation for the factor-price frontier in terms of the consumption good is

(7.7) $$W(r) = \frac{w}{p_3(r)} .$$

For an economy operating in equilibrium on this factor-price frontier, equilibrium prices in terms of the consumption good are

(7.8) $$P^e(r) = W(r)A_0\left[B-(1+r)A\right]^{-1}$$

and the price of the consumption good is $P_3^e(r) \equiv 1$ as in Hicks.

Using (7.6)-(7.8) to compute $P_3^e(r)$, we find that $P_3^e(r) \equiv 1$ is equivalent to the Hicks condition for equilibrium in the capital markets:

$$(7.9) \qquad \frac{0 - W(r)a_1}{1+r} + \frac{b_2 - W(r)a_2}{(1+r)^2} + \frac{1 - W(r)a_3}{(1+r)^3} = 0 = k_0 \; .$$

Similarly, we can show that the value of the production process at the beginning of period 2 is

$$(7.10) \qquad P_2^e(r) = k\left[2, r, 2; W(r)\right]$$

where the right-hand-side of (7.10) is defined by equation (A.1) in Appendix A.[11] Written out in full equation (7.10) may be expressed as

$$(7.11) \qquad W(r)a_1(1+r) + W(r)a_2 - b_2 = \frac{1 - W(r)a_3}{1+r} \; .$$

The left-hand-side is the net cost of producing a one-year-old machine. The right-hand-side is the present discounted value of the future revenue it can produce, which is also the value of the production process at the beginning of period 2. In equilibrium these two values must be equal.

If $P_2^e(r) \le 0$, as it will be if $a_3 \ge \frac{1}{W(r)}$, then Activity 3 is shut down and the Hicks production process is truncated. This example also provides another counterexample to Austrian Property 5 stated in the previous Section 6. Suppose $a_3 = 1$ and $W(0) > 1$ so that Activity 3 is not profitable at low interest rate. Since $\frac{dW(r)}{dr} < 0$, there is some interest rate r_1 for which $W(r_1) = 1$. Activity 3 becomes profitable at high interest rates with $1 < \frac{1}{W(r)}$ for $r > r_1$. Thus the optimal length of the production process increases with an *increase* in the interest rate.[12]

We conclude that the Hicks neo-Austrian approach is but a special case of the von Neumann approach. Additional details and discussion are contained in [Burmeister, 1974]. Kurz and Salvadori [1995] provide superb analyses of more complex technologies, including, for example, models with fixed capital, joint production, jointly utilized machines, and land.

8. The Necessary and Sufficent Condition for a "Well-Behaved" Economy Across Steady-State Equilibria

Once it is recognized that nothing of economic substance is to be gained from the neo-Austrian approach to capital theory, we turn to a more general question:

> Under what circumstances will an economy be "well-behaved" in the sense that steady-state per capita consumption always rises (falls) with decreases in the rate of interest, provided $r > g$ ($r < g$)?

The answer to this question has a surprisingly simple characterization. Provided joint production of final consumption goods is excluded, it has been shown that a neoclassical economy with a single consumption good and *N* different types of heterogeneous capital goods is "well-behaved" in the above sense if, and only if,

(8.1) $$\sum_{i=1}^{N} p_i(r)\frac{dk_i(r)}{dr} < 0 \quad \text{for all feasible } r,$$

where $k_i(r)$ denotes the steady-state per capita quantity of the i^{th} capital good at interest rate *r*. This result was first proved by Burmeister and Dobell [Burmeister and Dobell, 1970, Theorem 7, pp. 286-287]; see also [Burmeister and Turnovsky, 1972].

Denoting the value of capital by $v(r) = \sum_{i=1}^{N} p_i(r)k_i(r)$, we have:

(8.2) $$\text{Total Wicksell Effect } = \frac{dv(r)}{dr},$$

(8.3) $$\text{Price Wicksell Effect } = \sum_{i=1}^{N}\frac{dp_i(r)}{dr}k_i(r),$$

and

(8.4) $$\text{Real Wicksell Effect } = \sum_{i=1}^{N} p_i(r)\frac{dk_i(r)}{dr}.$$

If prices are in terms of the wage rate, the Price Wicksell effect is always positive. And it can be so positive that in outweighs a negative Real Wicksell effect. But this does not matter. No matter how the per capita value of capital behaves, it is the sign of the Real Wicksell Effect that is of fundamental economic significance. Another way to see this fact is to consider new prices measured in terms of the consumption good. The **sign** of the Real Wicksell Effect is not influenced by this change (or any other change), although the sign of the Price Wicksell Effect and hence of the Total Wicksell Effect may be altered.

This result generalizes to non-differentiable technologies such as our generalized von Neumann model. Suppose that Technique A is viable for $g < r < r^S$, Technique B is viable for $r > r^S > g$, and both are viable at the switching point $r = r^S$ with $p^A = p^B = p$, with p is a row vector in terms of the single consumption good. Define the change at the switch point in per capita consumption $\Delta c \equiv c^A - c^B$, and define the corresponding change in per capita capital stocks by the column vector $\Delta k \equiv k^A - k^B$. It can then be shown that per capita consumption rises as the rate of interest falls from above r^S to below r^S and there is a switch from Technique A to Technique B if, and only if,

(8.5) $$p\Delta k < 0 \ ,$$

which is a generalization of (8.1) to non-differentiable technologies. Proofs are contained in [Burmeister, 1976] and Burmeister [1974, p. 453].

We see, therefore, that the Hicks Simple Profile assumption plays the role of a sufficient condition ensuring that the Real Wicksell Effect in his model (if the intermediate capital goods are properly defined as in our von Neumann generalization) is always negative.

But, historical concerns aside, why should we even care about whether or not an economy is "well behaved" across alternative steady-state equilibria? After all, in Section 4 we argued that comparisons of steady-state equilibria are not of great economic interest because they do not represent the viable alternatives open to an economy.

I can think of only one reason. Sometimes it may be possible to interpret economic data as having been generated from an economy in alternative steady-state equilibria. In such cases it would be comforting if a rigorous theoretical foundation could be found for the existence of an aggregate production function of the type so often used, or misused, in econometric work. In a long-term research project now stretching out over more than five years, I have established that an index of aggregate capital and a corresponding well-behaved aggregate production function exist across steady-state equilibra if, and only if, the Real Wicksell Effect is negative at all feasible interest rates [Burmeister, 2000].[13]

Ironically, one of the few known sufficient conditions for negative Real Wicksell Effects is a generalization of the Marx Equal Organic Composition of Capital condition; see [Burmeister, 2001].[14]

9. Dynamics and Technological Change

Normally one thinks of a growth path as starting from arbitrary initial conditions. Instead, Hicks restricts his attention to paths that start in some steady-state equilibrium, and he calls the resulting dynamic path a *Traverse*. The economy moves from this initial steady state because there is some technological change that shifts the factor-price frontier outward and consequently makes a new process more profitable at the previous rate of interest. He considers two distinct cases: *The Fixwage Path* [Hicks, 1973, Chapter VIII, pp. 89-99] and *The Full Employment Path* [Hicks, 1973, Chapter IX, pp. 100-109]. Along a Fixwage Path, the real wage rate is given, and hence the rate of interest is determined from the factor-price frontier. Along a Full Employment Path, the interest rate is determined by the condition $r = \frac{g}{s}$ where s is the fixed savings propensity out of profit income, so there is no maximizing behavior on the part of consumers. In both cases Hicks assumes that all production processes have the Simple Profile described in Footnote 7.

Now let asterisks denote the steady-state equilibrium before the technological change. Hicks then defines the index measuring technological change

(9.1) $$I(r) \equiv \frac{w(r)}{w^*(r)} ,$$

which he calls "… an Index of Improvement in Efficiency, *in one sense or another* [Hicks, 1973, p. 75, italics in the original]." Within this restrictive framework, Hicks is able to revisit the question of Ricardo on machinery [Ricardo, 1911, Chapter XXXI]. Ricardo claimed that the introduction of machinery could have an adverse effect on the total wage bill in the short run. Hicks is able to find an interpretation of his model for which this conclusion is correct [Hicks, 1973, pp. 98-99].[15]

However, as I pointed out in [Burmeister, 1974, pp. 436-437], the set of possible outcomes increases when more than one primary factor of production exists. For example, consider a Ricardo model with two primary factors of production, labor and land, and let the real rental rate for land (in terms of the single consumption good) be denoted by π. Then (again ruling out joint production and assuming constant returns to scale) there exists a factor-price surface

(9.2) $$r = \Gamma(w, \pi)$$

defined across steady-state equilibria. Moreover, the trade-off between w and π for given r is quasi-convex.[16] Accordingly, when "the introduction of machinery" shifts Γ outward from the origin, it is possible for the economy to settle down in a new steady state with a higher real wage rate, a lower interest rate, and a lower rental rate for land. Thus, without more being said about the model, one cannot rule out the possibility that an innovation helps workers at the expense of the owners of both capital and land.

The Hicks analysis of dynamic paths is further simplified by his assumption of "… *static expectations*—that the wage that is ruling at [the current time] is expected to remain unchanged, at least so long as the processes started at [the current time] are expected to continue [Hicks, 1973, p. 110]."[17] Given these extraordinarily strong assumptions about technology and behavior, and sometimes using yet additional assumptions, Hicks is able to prove that some paths converge to a new steady-state equilibrium. The analysis is tedious and full of details about the characteristics of the dynamic paths. If these details represented general economic properties, some of them might be of considerable economic interest. But they do not hold in general. They have been shown to hold only for very special and economically unappealing cases.

What do we know in general about the economic properties of the sort of dynamic paths considered by Hicks? Many results were established in the decade after ***Capital and Time*** was published. Bliss [1975] is a good starting point, and Burmeister [1980, Chapter 5] provides an introductory exposition with references to some of the more technical literature. One of the most important theorems to emerge is contained in two classic papers by Cass [1975a and 1975b] where he proved the necessary and sufficient conditions for consumption efficiency over an infinite time horizon.

Three results merit special mention because they concern the kind of economic questions in which Hicks was most interested:

First, consider the intertemporal production possibility frontier for a constant-returns-to-scale competitive economy:

(9.3) $$f\left(c_0, c_1, \ldots, c_T; k_0, k_T\right) = 0$$

where c_t denotes per capita consumption and the beginning of period t, k_0 is a vector of the N heterogeneous capital goods (per capita) at the beginning of the initial period, and k_T is a vector of the terminal capital stocks. Both k_0 and k_T are given. Then the interest rate over period t is given by

(9.4) $$r_t = -\left.\frac{\partial c_{t+1}}{\partial c_t}\right|_{f=0} - 1 \ .$$

If f is not differentiable, r_t is bounded by right- and left-hand partial derivatives.[18] This result stems from the work of Irving Fisher and is developed in many of Samuelson's writings. In particular, an interesting account of the connection between this correct result and the mistake that was revealed by the existence of reswitching, see [Samuelson, 1966, footnotes 6, 7, 8, and 9].

Second, a result similar to the first holds for the real wage rate. With a non-differentiable technology such as the generalized von Neumann model discussed here, at every point in time the technology and competitive equilibrium impose upper and lower bounds on the possible values of the real wage rate. Within these bounds—but *only* within these bounds—there is room for a theory of income distribution that depends, for example, on the power of labor unions. Some of these results are contained in [Burmeister, 1984].

And third, the steady-state equilibrium for dynamic models with more than one capital good is usually a saddlepoint. Therefore the models converge only if there is some economic mechanism for determining the proper initial conditions to put the economy on the stable manifold. Some of the original papers addressing this problem include [Burmeister, Caton, Dobell, and Ross, 1973], [Hahn, 1966], [Malinvaud, 1953], [Samuelson, 1967], [Shell and Stiglitz, 1968], [Burmeister and Graham, 1974], and [Brock, 1972].

10. Lack of Impact and Unresolved Questions

Capital and Time has had little enduring impact on the economics profession, as revealed by the citation count in Table 3 for the twenty-two years from 1980 to 2001. The reason, I believe, is not so much because of Hicks' neo-Austrian approach, but rather because the kind of analysis contained in *Capital and Time* became technically obsolete soon after its publication in 1973. New tools were developed, and it changed the kind of economic models that we build. Rational expectations should be used and the Hicks' assumption of static expectations is no longer acceptable. The technology specification should include stochastic shocks. Both producers and consumers should exhibit maximizing behavior under uncertainty. Technological change should not be exogenous, but rather should arise as the result of economic decisions. But these were not common features of economic models in 1973.

It appears that ***Capital and Time*** represents the continuation of a research agenda laid out by Hicks in the second edition of ***The Theory of Wages*** [Hicks, 1963].[19] For example, in the section "Wages, Interest, and Growth" [Hicks, 1963, pp. 363-372], he explored the relationship between the real wage rate and the interest rate and asked how an economy can move from one steady-state equilibrium to another. Of particular interest was the question, "Under what conditions are labor unions able to raise the real wage rate?"

In my opinion, therefore, Hicks probably had two primary objectives in writing *Capital and Time*:

1. To clarify the pure role of time in economics, without the complications of uncertainty, by providing an alternative to the standard production function that would help us to better understand the determination of real wage rates.
2. To establish the dynamic stability properties for models using his alternative neo-Austrian technology.

In pursuing these objectives, Hicks brought to bear the tools that were available to him at the time. This resulted in a model containing many features, as noted above, that simply are unacceptable by modern standards. Consequently while he made considerable progress toward achieving these two objectives, he fell far short of what today we would deem to be success.

But we should not be too harsh judging Hicks. The fundamental questions that might have been partially answered had Hicks achieved his objectives remain essentially unresolved today:

First, many fields of economics continue to use models with aggregate capital and an aggregate production function even though these have no rigorous theoretical foundation, except under extraordinarily restrictive assumptions. Other aspects of these modern models are often very sophisticated—but aggregate capital is a shaky logical foundation upon which to build. I do not mean to imply that these models are "wrong." All models are of necessity no more than a caricature of reality, and their usefulness depends upon whether or not they are able to shed light on interesting economic problems. By this criterion many models using aggregate capital are clearly a success. We will continue to see such models until someone discovers a better alternative, and that is how research should progress. But the Hicks neo-Austrian technology did not provide a better alternative.

Second, the dynamic stability properties of models containing more than one capital good have not been completely worked out. We know a lot about special cases, especially when there is maximizing behavior and rational expectations. But in general it is difficult to establish stability and to rule out cycles, even without the realistic complications introduced by uncertainty.

I conclude that despite Hicks' attempt to address questions of fundamental economic importance, he was unable to shed much light on them. And so, I am afraid, for the most part the economics profession has simply ignored *Capital and Time*. Yet it remains a delightful book to read, and one can still learn a lot of economics from it.

Citation Count

Year	*Capital and Time* [Hicks, 1973]	*Value and Capital* [Hicks, 1939]	*Foundations of Economic Analysis* [Samuelson, 1947]
2001	0	9	25
2000	3	9	36
1999	3	7	39
1998	1	10	29
1997	0	10	47
1996	2	9	26
1995	2	5	42
1994	1	6	40
1993	2	4	42
1992	0	4	43
1991	0	2	35
1990	2	8	50
1989	2	1	28
1988	0	7	44
1987	1	5	60
1986	3	9	35
1985	3	7	65
1984	4	9	57
1983	2	12	67
1982	1	10	43
1981	3	8	62
1980	1	7	63
Total	**36**	**158**	**978**
Average	**1.64**	**7.18**	**44.45**

Source: Social Science Citation Index (SSCI)—1978-present, via Web of Science web site at <http://webofscience.com/>.

Table 3

Appendix A

For completeness we sketch a proof of Hicks' Fundamental Theorem using our timing convention for inputs and outputs.

Step 1:

Given a real wage rate, the value of a production process at the beginning of period *t*, when that process is operated through to period *T*, is given by

$$k(t, r, T; w) \equiv \sum_{i=t}^{T} q_i (1+r)^{-(i+1-t)}, \quad t \le T \le n \ . \tag{A.1}$$

Taking the interest rate *r* as given for now, it is assumed that there exists some duration *T* for which the process is strictly viable:

$$k(0, r, T; w) > 0 \ . \tag{A.2}$$

Step 2:

If the process is operated for its optimal duration, Ω, the capital value at the beginning of period 0 is maximized and by (A.2) this value is positive:

$$k(0, r, \Omega; w) > 0 \ . \tag{A.3}$$

Also by definition of Ω

$$k\left(0, r, \Omega; w\right) > k\left(0, r, T; w\right) \quad \text{for all} \quad T \neq \Omega \tag{A.4}$$

and hence

$$k\left(0, r, \Omega; w\right) - k\left(0, r, \Omega + i; w\right) > 0, \quad i = 1, 2, \ldots, n - \Omega \ . \tag{A.5}$$

We observe that in the case $n > \Omega$

$$k\left(\Omega + 1, r, t; w\right) < 0, \quad t = \Omega + 1, \Omega + 2, \ldots, n \ . \tag{A.6}$$

Thus the capital values of the process at the beginning of period $\Omega + 1$ are negative when it is operated for durations $\Omega + 1$ or longer.

Step 3:

The trivial case $\Omega = 0$ is excluded. We then see from (A.4) that

(A.7) $$k(0, r, \Omega; w) - k(0, r, \Omega - i; w) > 0, \quad i = 1, 2, \ldots, \Omega - 1 .$$

Therefore

(A.8) $$k(t, r, \Omega; w) > 0, \quad t = 1, \ldots, \Omega$$

and from (A.3) this result extends to

(A.9) $$k(t, r, \Omega; w) > 0, \quad t = 0, 1, \ldots, \Omega .$$

Thus when the process is operated for its optimal duration, the capital values of the process at the beginning of each period $t = 0, 1, \ldots, \Omega$ are positive.

In particular,

(A.10) $$k(\Omega, r, \Omega; w) = \frac{q_\Omega}{1 + r} > 0 .$$

Step 4:

Differentiating (A.10) with respect to *r*, we see that

(A.11) $$\frac{d\left[k(\Omega, r, \Omega; w)\right]}{dr} < 0 .$$

Step 5:

Using the notation in this Appendix, equation (2.4) in the text may be written as

(A.12) $$k(t, r, \Omega; w) = \frac{q_t + k(t + 1, r, \Omega; w)}{1 + r}$$

or, equivalently,

(A.13) $$k(t - 1, r, \Omega; w) = \frac{q_{t-1} + k(t, r, \Omega; w)}{1 + r} .$$

Step 6:

Setting $t = \Omega$ in (A.13) and differentiating with respect to *r*, we obtain

(A.14) $$\frac{d\left[k(\Omega-1,r,\Omega;w)\right]}{dr} = -\frac{k(\Omega-1,r,\Omega;w)}{(1+r)^2} + \frac{d\left[k(\Omega,r,\Omega;w)\right]}{dr} < 0$$

because of (A.9) and (A.11). Continuing in this manner, we see that

(A.15) $$\frac{d\left[k(t,r,\Omega;w)\right]}{dr} < 0, \quad t = 0,1,\ldots,\Omega .$$

Step 7:

At some higher interest rate $r + \Delta r$, it is possible that

(A.16) $$k(T, r+\Delta r, \Omega; w) < 0 \quad \text{for some} \quad T < \Omega .$$

In this case at the new higher interest rate there is a new optimal duration $\Omega^* < \Omega$ for which

(A.17) $$k(t, r+\Delta r, \Omega^*; w) > 0, \quad t = 0,1,\ldots,\Omega^* .$$

In view of (A.8) we see that

(A.18) $$k(t, r, \Omega^*; w) > 0, \quad t = 0,1,\ldots,\Omega^* < \Omega .$$

Hence the previous argument shows that all of these must fall when *r* rises to $r + \Delta r$.

Step 8:

We conclude that the present discounted value of the production process, which Hicks denotes simply by k_0, is initially positive and falls continuously with increasing *r*, even if the duration becomes shorter with increasing *r*. Thus there exists a unique $r = r^0$ and a corresponding optimal duration Ω^0 such that the present discounted value of the production process at the beginning of period 0 is equal to 0:

(A.19) $$k(0, r^0, \Omega^0; w) = 0 .$$

The interest rate $r = r^0$ is the unique internal rate of return for the production process at the wage rate *w*.

Moreover, if (A.19) holds and r^0 is increased to $r^1 = r^0 + \Delta r$ so that

(A.20) $$k\left(0, r^1, \Omega^1; w\right) < 0 \ ,$$

there exists some $w^1 < w$ such that

(A.21) $$k\left(0, r^1, \Omega^1; w^1\right) = 0$$

where the optimal duration at interest rate r^1 is $\Omega^1 \leq \Omega^0$. This establishes that the factor-price curve for the production process is downward sloping.

Appendix B

Here we sketch a generalization of the original von Neumann model [von Neumann, 1938, 1945-46] to allow for labor as a primary factor of production.

There are *m* different activities for producing *n* different commodities $(m \leq n \text{ or } m \geq n)$. Activity *j* operated at the unit intensity level requires a labor input a_{0j} and a vector of commodity inputs $(a_{1j}, \ldots, a_{nj})$ to produce a vector of commodity outputs $(b_{1j}, \ldots, b_{nj})$. We define the vector of labor requirements

(B.1) $$A_0 = (a_{01}, \ldots, a_{0m}) \ ,$$

the input matrix

(B.2) $$A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix},$$

and the output matrix

(B.3) $$B = \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix}.$$

The column vector

(B.4) $$x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

give the intensity levels at which each of the *m* activities are operated. Finally, the vector

(B.5) $$C = \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix}$$

represents the consumption of commodities.

Labor grows at the exogenous rate $g \geq 0$. The system is capable of growth at rate g if

(B.6) $$\left(1+g\right)Ax \leq Bx - C \quad \text{with} \quad C \geq 0,\ C \neq 0 \quad \text{and} \quad x \geq 0,\ x \neq 0\ .$$

The row vector the n commodity prices is

(B.7) $$p = \left(p_1,\ \ldots,\ p_n\right) \quad \text{with the normalization} \quad \sum_{i=1}^{n} p_i = 1\ ,$$

the wage rate is $w \geq 0$, and the steady-state rate of interest is $r \geq 0$. A steady-state equilibrium at a given value of r is possible if there is a solution to the dual von Neumann price system satisfying

(B.8) $$wA_0 + \left(1+r\right)pA \geq pB \quad \text{with} \quad w \geq 0,\ p \geq 0,\ p \neq 0\ .$$

If the cost of operating an activity exceeds its revenue, that activity is shut down (is operated at a zero intensity level):

(B.9) $$x_j = 0 \quad \text{if} \quad wa_{0j} + \left(1+r\right)\sum_{i=1}^{n} p_i a_{ij} > \sum_{i=1}^{n} p_i b_{ij}, \quad j = 1, \ldots, m\ .$$

If an activity is operated at a positive intensity level, then revenue must exactly cover cost:

(B.10) $$wa_{0j} + \left(1+r\right)\sum_{i=1}^{n} p_i a_{ij} = \sum_{i=1}^{n} p_i b_{ij} \quad \text{if} \quad x_j > 0, \quad j = 1, \ldots, m\ .$$

Similarly, the price of a commodity is zero if it is in excess supply:

(B.11) $$p_i = 0 \quad \text{if} \quad \left(1+g\right)\sum_{j=1}^{m} a_{ij} x_j < \sum_{i=1}^{n} b_{ij} x_j - C_i, \quad j = 1, \ldots, m\ .$$

And if a commodity has a positive price, its supply and demand are equal:

(B.12) $$\left(1+g\right)\sum_{j=1}^{m} a_{ij} x_j = \sum_{i=1}^{n} b_{ij} x_j - C_i \quad \text{if} \quad p_i = 0, \quad i = 1, \ldots, n\ .$$

We denote employed labor by $L = A_0 x$. Then by combining (B.9)—(B.12) we see that in a steady-state equilibrium

(B.13) $$\left(1+g\right)pAx + pC = pBx = wL + (1+r)Ax\ .$$

Denoting the per capita value of capital by $v = \frac{pAx}{L}$ and the per capita value of consumption by $pc = \frac{pC}{L}$, equation (B.13) may be rewritten as

(B.14) $$pc = w + (r - g)v \ .$$

The latter may also be written in the more familiar form

(B.15) Consumption + Net Investment = $pc + gv$ = Wages + Profits = $w + rv$.

This is a generalization of the result derived by Hicks in his "Social Accounting" chapter [Hicks, 1973, pp. 28-36].

References

Akerlof, G. A. "The Market for 'Lemons': Quality Uncertainty and the Market Mechanism," ***Quarterly Journal of Economics,*** August 1970, 84(3), pp. 488-500.

Arrow, K. J. and Levhari, D. "Uniqueness of the Internal Rate of Return with Variable Life of Investment," ***Economic Journal*****,** Sept. 1969, 79, pp. 560-566.

Bliss, Christopher. ***Capital Theory and the Distribution of Income.*** Amsterdam: North-Holland, 1975.

Brock, W. A. "On Models of Expectations that Arise From Maximizing Behavior of Economic Agents over Time," ***Journal of Economic Theory,*** Dec. 1972, 5(3), pp. 348-376.

________. "Some Results on the Uniqueness of Steady States in Multisector Models of Optimum Economic Growth When Future Utilities Are Discounted," ***International Economic Review,*** Oct. 1973, 14(3), pp. 535-559.

Bruno, M. "Fundamental Duality Relations in the Pure Theory of Capital and Growth," ***Review of Economic Studies,*** Jan. 1969, 36(l), No. 105, pp. 39-53.

________, Burmeister, E., and Sheshinski, E. "The Nature and Implications of the Reswitching of Techniques," ***Quarterly Journal of Economics,*** Nov. 1966, 80(4), pp. 526-553.

_______, _________, and ________. "The Badly Behaved Production Function: Comment," ***Quarterly Journal of Economics,*** August 1968, 82, pp. 524-525.

Burmeister, Edwin. "Aggregate Production Functions," Draft Working Paper, Department of Economics, Box 90097, Duke University, Durham, NC 27708-0097, May 2000.

______. "Equal Organic Composition of Capital and Regularity," Draft Working Paper, Department of Economics, Box 90097, Duke University, Durham, NC 27708-0097, May 2001.

______. ***Capital Theory and Dynamics***. New York: Cambridge University Press, 1980.

______. "On a Theorem of Sraffa," ***Economica,*** N.S., Feb. 1968, 35, pp. 83-87.

______. "Real Wicksell Effects and Regular Economies," in ***Essays in Modern Capital Theory***, (M. Brown, K. Sato. and P. Zarembka, Editors), Amsterdam: North-Holland, 1976.

______. “The Factor-Price Frontier and Duality with Many Primary Factors,” ***Journal of Economic Theory***, Vol. 12, No. 3, pp. 496-503, June 1976.

______. “Sraffa, Labor Theories of Value, and the Economics of Real Wage Rate Determination,” ***Journal of Political Economy***, June 1984.

______. “Synthesizing the Neo-Austrian and Alternative Approaches to Capital Theory: A Survey,” ***Journal of Economic Literature***, June 1974, 12(2), pp. 413-456.

_______, Caton, C., Dobell, A, R. and Ross, S. A. "The 'Saddlepoint Property' and the Structure of Dynamic Heterogeneous Capital Good Models," ***Econometrica,*** Jan. 1973, 41(1), pp. 79-95.

______ and Dobell, A. R. ***Mathematical Theories of Economic Growth***. New York: Macmillan, 1970.

_______ and Graham, D. A. "Multi-Sector Economic Models with Continuous Adaptive Expectations," ***Review of Economic Studies,*** July 1974, 41(3), pp. 323-336.

______ and Kuga, K. "The Factor-Price Frontier, Duality and Joint Production," ***Review of Economic Studies,*** Jan. 1970, 37(109), pp. 11-19.

______ and ________. "The Factor-Price Frontier in a Neoclassical Multi-Sector Model," ***International Economic Review,*** Feb. 1970, 11(l), pp. 162-174.

_______ and Sheshinski, E. "A Non-substitution Theorem in a Model with Fixed Capital," ***Southern Economic Journal,*** Jan. 1969, 35(3), pp. 273-276.

_______ and Taubman, P. "Labour and Non-labour Income Saving Propensities," ***Canadian Journal of Economics***, Feb. 1969, 2(l), pp. 78-89.

______ and Turnovsky, S. J. "Capital Deepening Response in an Economy with Heterogeneous Capital Goods," ***American Economic Review***, Dec. 1972, 62(5), pp. 842-853.

_______ and Leland B. Yeager. “Continuity and Capital Reversal: Reply,” ***Economic Inquiry***, January 1978.

Cass, David. "On the Wicksellian Point-Input,Point-Output Model of Capital Accumulation: A Modern View (or Neoclassicism Slightly Vindicated)," ***Journal of Political Economy***, 81(1), pp. 71-97, January/February 1973.

______. “Distinguishing Inefficient Competitive Growth Paths: A Note on Capital Overaccumulation and Rapidly Diminishing Future Value of Consumption in a Fairly General Model of Capitalistic Production,” ***Journal of Economic Theory***, Volume 4, April 1972a.

______. “On Capital Overaccumulation in the Aggregative, Neoclassical Model of Economic Growth: A Complete Characterization,” ***Journal of Economic Theory***, Volume 4, April 1972b.

Dorfman, R., Samuelson, P. A. and Solow, R. M. ***Linear Programming and Economic Analysis***. New York: McGraw-Hill, 1958.

Flemming, J. S. and Wright, J. F. "Uniqueness of the Internal Rate of Return: A Generalisation," ***Economic Journal,*** June 1971, 81(322), pp. 256-263.

Hagemann, Harald. “Employment and Machinery,” pp. 200-224 in ***The Legacy of Hicks*** (Harald Hagemann and O. F. Hamouda, Editors), London: Routledge, 1994.

______ and Heinz D. Kurz. “The Return of the Same Truncation Period and Reswitching of Techniques in Neo-Austrian and More General Models,” ***Kyklos***, Volume 29, pp. 678-708, 1976.

Hahn, F. H. "Equilibrium Dynamics with Heterogeneous Capital Goods," ***Quarterly Journal of Economics***, Nov. 1966, 80(4), pp. 633-646.

Harcourt, G. C. "Some Cambridge Controversies in the Theory of Capital," ***Journal of Economic Literature***, June 1969, 7(2), pp. 369-405.

Heller, W. P. "Disequilibrium Dynamics of Competitive Growth Paths," ***Review of Economic Studies,*** Oct. 1971, 38(4), No. 116, pp. 385-400.

Hicks, John R. ***Value and capital***. New York and London: Oxford University Press, 1939.

________. ***Value and Capital***, Second Edition, Oxford: Clarendon Press, 1946.

_______ . ***Theory of Wages***. London: Macmillan, 1932.

_______ . ***Theory of Wages***. 2nd edition. London: Macmillan, 1963.

_______. ***Capital and Growth***. New York and London: Oxford University Press, 1965.

_______. **A T*heory of Economic History***. New York and London: Oxford University Press, 1969.

_______. ***Capital and time: A Neo-Austrian Theory***. New York and London: Oxford University Press, 1973.

Karlin, S. ***Mathematical Methods and Theory in Games, Programming,and Economics***. Vols. I and II. Reading, Mass.: Addison-Wesley, 1959.

Kurz, Heinz D. and Neri Salvadori. ***Theory of Production: A Long-Period Analysis***, Cambridge: Cambridge University Press, 1995.

Malinvaud, E. "Capital Accumulation and Efficient Allocation of Resources," ***Econometrica,*** April 1953, 21(2), pp. 233-268.

______. "Efficient Capital Accumulation: A Corrigendum," ***Econometrica,*** July 1962, 30(3), pp. 570-573.

Mirrlees, J. A. "The Dynamic Nonsubstitution Theorem," ***Review of Economic Studies***, Jan. 1969, 36(l), pp. 67-76.

Morishima, M. ***Equilibrium, Stability, and Growth:* A M*ulti-Sectoral Analysis***. Oxford: Clarendon Press, 1964.

______. ***Theory of Economic Growth***. Oxford: Clarendon Press, 1969.

Muth, J. F. "Rational Expectations and the Theory of Price Movements," ***Econometrica,*** July 1961, 29(3), pp. 315-335.

von Neumann, J. "Über ein Ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes," ***Ergebuisse eines Mathematischen Seminars***. Edited by Menger, K., Morgenstern, G., translator. Vienna: 1938. English translation: "A Model of General Economic Equilibrium," ***Review of Economic Studies,*** 1945-46, 13, pp. 1-9. Reprinted in ***Readings in Mathematical Economics***. Edited by Peter Newman. Vol. II. Baltimore: Johns Hopkins University Press, 1968. Pp. 221-229.

"Paradoxes in Capital Theory: A Symposium," ***Quarterly Journal of Economics***, Nov. 1966, 80(4), pp. 503-583.

Pigou, A. C. ***The Economics of Welfare***. 4th edition. London: Macmillan, 1932.

Ricardo, D. ***The Principles of Political Economy and Taxation***. London: [1911], 1960.

Samuelson, Paul A. ***Economics: An Introductory Analysis***, Sixth Edition, New York: McGraw-Hill Book Company, 1964.

______. ***Foundations of Economic Analysis***, Cambridge, MA: Harvard University Press, 1947.

______. ***Foundations of Economic Analysis***, 1983 Enlarged Edition, Cambridge, MA: Harvard University Press, 1983.

______. "Indeterminacy of Development in a Heterogeneous-Capital Model with Constant Saving Propensity." In ***Essays on the Theory of Optimal Economic Growth***. Edited by Karl Shell. Cambridge: M.I.T. Press, 1967. Pp. 219-231.

________. "A Modern Treatment of the Ricardian Economy: I. The Pricing of Goods and of Labor and Land Services," ***Quarterly Journal of Economics***, Feb. 1959, 73(l), pp. 1-35. Reprinted in ***The Collected Scientific Papers of Paul A. Samuelson***. Cambridge, Mass.: M.I.T. Press, 1966.

______. "A Modern Treatment of the Ricardian Economy: II. Capital and Interest Aspects of the Pricing Process," ***Quarterly Journal of Economics***, May 1959, 73(2), pp. 217-231. Reprinted in ***The Collected Scientific Papers of Paul A. Samuelson***. Cambridge, Mass.: M.I.T. Press, 1966.

______. "Parable and Realism in Capital Theory: The Surrogate Production Function," ***Review of Economic Studies***, June 1962, 29(3), No. 80, pp. 193-206. Reprinted in ***The Collected Scientific Papers of Paul A. Samuelson***. Cambridge, Mass.: M.I.T. Press, 1966.

______. "A Summing Up," ***The Quarterly Journal of Economics***, LXXX, Number 4, pp. 568-583, November 1966.

______. "Wages and Interest: A Modern Dissection of Marxian Models," ***American Economic Review***, Vol. XLVII, No. 6. pp. 884-912, December 1957.

______. "Steady-State and Transient Relations: A Reply on Reswitching," ***The Quarterly Journal of Economics***, Vol. 89, pp. 40-47, 1975.

Sen, A. "Minimal Conditions for Monotonicity of Capital Value" ***Journal of Economic Theory***, Vol. 11, December 1975, pp. 340-355.

Shell, K. and Stiglitz, J. E. "The Allocation of Investment in a Dynamic Economy," ***The Quarterly Journal of Economics***, Nov. 1968, 82(4), pp. 537-560.

Sraffa, P. ***Production of Commodities by Means of Commodities: Prelude to a Critique of Economic Theory***. Cambridge: Cambridge University Press, 1960.

Stiglitz, J. E. "Non-Substitution Theorems with Durable Capital Goods," ***Review of Economic Studies***, Oct. 1970, 37(4), No. 112, pp. 543-570.

von Weizsäcker, C. C. "Bemerkungen zu einem 'Symposium' Über Wachstrumstheorie und Produnktionfunktionen," ***Kyklos***, 1963, 16, pp. 438-47.

______. ***Steady State Capital Theory***. New York: Springer-Verlag, 1971.

Wicksell, K. ***Lectures on Political Economy***. Translated from Swedish by E. Classen. 2 Volumes. London: Routledge & Kegan Paul, 1934; Clifton, N.J.: Augustus M. Kelley, 1967.

Yeager, Leland B. "Towards Understanding Some Paradoxes in Capital Theory," ***Economic Inquiry***, pp. 313-346, 1976.

Footnotes

[1] An earlier draft of this paper was presented to a joint meeting of the Economic Theory and the History of Political Economy Workshops at Duke University, and I am grateful for the comments received there. My special thanks goes to Neil De Marchi for the many improvements that he suggested and to Daniel A. Graham for his help and encouragement.

[2] The list of topics [Burmeister, 1974, pp. 413-414] was:

(1) truncation of production process and the uniqueness of the internal rate of return;
(2) factor-price curves and the factor-price frontier;
(3) reswitching (both as usually defined and along a dynamic path);
(4) duality results;
(5) determination of relative factor shares;
(6) complications arising from joint production (including an example showing why the nonsubstitution theorem is invalid when certain types of joint production exist);
(7) technological change and the relationship of Hicks' old classification, his new classification, and Harrod neutrality;
(8) a clarification of the famous "Ricardo on Machinery" dispute;
(9) dynamic paths and stability (or the Traverse) when problems of uncertainty are circumvented;
(10) two numerical examples illustrating full-employment transitions;
(11) a simple demonstration that the neo-Austrian method is a special case of the more general von Neumann approach;
(12) an interpretation of a neo-Austrian example without joint production as a specialized Leontief-Sraffa model;
(13) the problem of substitution and a brief explanation of i) stability results from other sectoral models, and ii) the similarity of the strong assumptions which are required for convergence;
(14) a very short summary of the subjects which traditionally have been controversial in capital theory (including a paradox revealed by the reswitching controversy); and
(15) a generalized von Neumann model with consumption and a primary factor (presented in the Appendix).

[3] Hicks assumes that inputs and outputs are both valued at the beginning of the week [Hicks, 1973, p. 20]. The more conventional assumption used here is that labor is paid and revenue from output is received at the end of the period. Accordingly, Hicks' capital values are related to ours by $k_t^{Hicks} = k_t\left(1+r\right)$. The critical observation is that the equilibrium conditions $k_0^{Hicks} = 0$ and $k_0 = 0$ both define the exactly same relationship between the real wage rate and the interest rate.

[4] The exactly analogous relation given Hicks' timing assumptions is $k_t^{Hicks} = q_t^{Hicks} + k_{t+1}^{Hicks} R^{-1}$; see [Hicks, 1973, equation (2.1), p. 20].

[5] Similarly in Appendix A we introduce the following more precise notation for k_t: Given a real wage rate, the value of a production process at the beginning of period *t*, when that process is operated through to period *T*, is given by $k(t,r,T;w) \equiv \sum_{i=t}^{T} q_i (1+r)^{-(i+1-t)}, \quad t \le T \le n \cdot$

[6] In earlier work [Hicks, 1965] he refers to the same relationship as the *wage-interest curve*.

[7] Using the notation in Appendix A, define the implicit function $G(w,r) \equiv k(0,r,\Omega;w) = 0$. Then

$$\frac{dr}{dw} = -\frac{\partial G/\partial w}{\partial G/\partial r} = -\frac{\sum_{i=0}^{\Omega}(-a_i)(1+r)^{i+1}}{\partial k(0,r,\Omega;w)/\partial r} < 0$$

because (1) by assumption $a_i \geq 0$ and at least one labor input is positive, and (2) we have already shown in Appendix A that, for a given real wage rate, the capital value of the process at the beginning of period 0 always falls with a rise in the interest rate.

[8] Earlier Hicks gave a different definition for the "average period of production" and claimed that it always increased with decreases in the rate of interest; see [Hicks, 1946, Appendix to Chapter XVII]. This claim was shown to be false by Samuelson [Samuelson, 1947, p. 188]. Subsequently work by Hicks reveals agreement about these issues [Hicks, 1965, Chapter XIII].

[9] A Simple Profile has input a_c and output 0 for weeks $0, 1, \ldots, m-1$ and it has input a_u and output 1 for weeks $m, m+1, \ldots, m+n-1$.

[10] As far as I know, Leland was allowed to keep his prize.

[11] Some caution is required because our subscripts for the von Neumann activities are one more than the Hicks subscripts. This is because our Activity 1 is associated with the Hicks production period 0, etc.

[12] The set of parameters generating this example in which Activity 3 is shut down when $r < 1$ is $a_1 = a_2 = a_3 = 1$, $b_2 = 3$, and $w = 0.5$. However, if a_3 is reduced to 0.5, then the production process is never truncated because $W(0) < 2$.

[13] My more ambitious goal is to compute the explicit functional forms for both aggregate capital and the aggregate production function for specific economies with heterogeneous capital goods, but this turns out to be an extraordinarily complex computational task. With each improvement in the Maple engine, I have been able to make a little more progress, but there is still a long way to go. Meanwhile—and Bob Solow told me that this news came as no surprise at all to him—I can report that the Cobb-Douglas functional form is often an excellent approximation for the true aggregate production function.

[14] The Equal Organic Composition of Capital condition for a no-joint production neoclassical technology is that, across steady-state equilibria, the ratio of the wage bill to the value of capital be the same function of the interest rate for every industry.

[15] Those interested in this issue should not miss reading [Samuleson, 1959] and [Hagemann, 1994].

[16] These results, along with the associated duality relationships, are proved in [Burmeister, 1976]. Also see [Samuelson, 1975] where similar results were first stated without proof.

[17] Also see the discussion of static expectations on p. 56.

[18] It is implicitly assumed that an alternative feasible path exists which differs from the original competitive path $\{c_t, k_t\}_{t=0}^{T}$ only by Δc_t and Δc_{t+1}. For some technologies such an alternative path may not always exist, and then more complex methods are required to establish bounds on the interest rate.

[19] I am indebted to my colleague Neil De Marchi for this observation.