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Roy Model of Self-Selection: General Case

Results drawn on Heckman and Sedlacek *JPE*, 1985 and Heckman and Honoré, *Econometrica*, 1986.

Two-sector model in which:

Agents are income maximizers, i.e., agent works in sector in which has highest income.

Mobility between sectors is costless, but they can work in only one sector (sector 1 or sector 2).

Each sector requires sector-specific task and agents have two skills T_1 and T_2.

Assume aggregate skill distribution given, i.e., short-run model. (No investment possibilities to change skills.)

Prices for skills are assumed to be known by agents at time of making sectoral choice decision. (Certainty of prices not crucial.)

T_i denotes amount of sector i task an agent can perform.

π_i is price or return to worker for working in sector i, ($\pi_i > 0$). (No capital in this model.)

Continue with the normality assumption of original Roy Model, i.e.,

$$\begin{pmatrix} \ln T_1 \\ \ln T_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \Sigma \right) \tag{1}$$

So that the log wage for working in Sector i given by:

$$\ln W_i = \ln \pi_i + \ln T_i . \tag{2}$$

so that

$$\begin{aligned}\ln W_1 &= \ln \pi_1 + \mu_1 + U_1 \\ \ln W_2 &= \ln \pi_2 + \mu_2 + U_2\end{aligned}$$

where (U_1, U_2) is mean zero normal vector.

The Agent works in Sector 1 *iff:*

$$W_1 = \pi_1 T_1 > \pi_2 T_2 = W_2 \tag{3}$$

or

$$\begin{aligned}\ln W_1 &> \ln W_2 \\ \ln \pi_1 + \mu_1 + U_1 &> \ln \pi_2 + \mu_2 + U_2 \\ U_1 - U_2 &> \ln(\pi_2/\pi_1) + \mu_2 - \mu_1\end{aligned} \quad . \tag{4}$$

So that proportion of population working in sector 1 given by proportion of population for which:

$$T_1 > \frac{\pi_2}{\pi_1} T_2. \tag{5}$$

Then it follows that

$$\Pr(i) = P(\ln W_i > \ln W_j) = \Phi(c_i) \tag{6}$$

$i,j = 1, 2$, $i \neq j$, where $c_i = \left(\ln(\pi_i/\pi_j) + \mu_i - \mu_j\right)/\sigma^*$, $\sigma^* = \sqrt{\operatorname{var}(U_1 - U_2}$ and $\Phi(\cdot)$ is the CDF for a standard normal random variable.

It follows that the *observed wage* in sector i is:

$$E\left(\ln W_i \mid \ln W_i > \ln W_j\right) = \ln \pi_i + \mu_i + \frac{\sigma_{ii} - \sigma_{ij}}{\sigma^*}\lambda\left(c_i\right), \quad i, j = 1,2, i \neq j, \tag{7}$$

where

$$\lambda\left(c\right) = \frac{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}c^2\right)}{\Phi\left(c\right)}$$

is a convex monotone decreasing function of c with $\lambda\left(c\right) \geq 0$ and $\lim_{c\to\infty}\lambda\left(c\right) = 0$, $\lim_{c\to-\infty}\lambda\left(c\right) = \infty$. Furthermore, the variance of log wages *observed* in sector i is:

$$\operatorname{var}\left(\ln W_i \mid \ln W_i > \ln W_j\right) = \sigma_{ii}\left\{\rho_i^2\left[1 - c_i\lambda\left(c_i\right) - \lambda^2\left(c_i\right)\right] + \left(1 - \rho_i^2\right)\right\},\ i \neq j, \tag{8}$$

where $\rho_i = \operatorname{corr}\left(U_i, U_i - U_j\right), i \neq j = 1,2.$

NOTE: Variance of log of observed wages never exceeds σ_{ii}, the population variance, as the term in { } in (8) is always ≤ 1. Thus, sectoral variances always *decrease* with increased selection (see Heckman & Sedlacek or Heckman & Honoré).

It follows that the mean *observed* level of log skills in a sector is given by

$$E\left(\ln T_1 \mid \ln W_1 > \ln W_2\right) = \mu_1 + \frac{\sigma_{11} - \sigma_{12}}{\sigma^*}\lambda\left(c_1\right), \tag{9}$$

$$E\left(\ln T_2 \mid \ln W_2 > \ln W_1\right) = \mu_2 + \frac{\sigma_{22} - \sigma_{12}}{\sigma^*}\lambda\left(c_2\right). \tag{10}$$

What is the Nature of Distribution of Skills and Earnings under Self-Selection?

$$E\left(\ln T_1 \mid \ln W_1 > \ln W_2\right)\begin{bmatrix} > \\ = \\ < \end{bmatrix}\mu_1 \text{ as } \left(\sigma_{11} - \sigma_{12}\right)\begin{bmatrix} > \\ = \\ < \end{bmatrix}0, \tag{11}$$

where σ_{11}-σ_{12} = cov(U_1,[U_1-U_2]) and

$$E\left(\ln T_2 \mid \ln W_2 > \ln W_1\right)\begin{bmatrix} > \\ = \\ < \end{bmatrix}\mu_2 \text{ as } \left(\sigma_{22} - \sigma_{12}\right)\begin{bmatrix} > \\ = \\ < \end{bmatrix}0, \tag{12}$$

1. If $\sigma_{12} = 0$, i.e., endowments of sector-specific skills are uncorrelated, self-selection always leads to the mean of lnT_1 employed in sector 1 to *exceed* μ_1 and to the mean of lnT_2 employed in sector 2 to *exceed* μ_2.

2. If $\sigma_{11} - \sigma_{12} > 0$, self-selection also always leads to the mean of lnT_1 employed in sector 1 to *exceed* μ_1. The relationship between the mean of lnT_2 employed in sector 2 and μ_2, depends on the sign of ($\sigma_{22} - \sigma_{12}$).

   If $\sigma_{22} - \sigma_{12} > 0$, then the mean of ln$T_2$ employed in sector 2 *exceeds* μ_2. This is referred to as the case of ***comparative advantage*** or ***non-hierarchical sorting***, i.e., self-selection on income leads to workers sorting into sectors in which they have a comparative advantage in terms of their skills. Note that this case is more likely to occur when $\sigma_{12} < 0$, i.e., when the sector-specific skills are *negatively correlated*.

   We consider the case where $\sigma_{22} - \sigma_{12} < 0$ in the next case.

3. If $\sigma_{11} - \sigma_{12} < 0$, self-selection always leads to mean of lnT_1 employed in sector 1 to fall *below* μ_1. At the same time, the mean of lnT_2 must lie *above* μ_2.[1] Thus, this somewhat "unusual" case, i.e., that self-selection actually reduces the mean skills in the selected sector, *can only occur in one sector*, *but not both*. Note that this case requires that σ_{12} be *sufficiently positive*. This is referred to as the case of ***absolute advantage*** or ***hierarchical sorting***, i.e., agents tend to have high or low skills in *both* sectors. More on this situation below.

[1] This is result of fact that Σ is positive definite, which implies $\sigma_{11} + \sigma_{22} - 2\sigma_{12} \geq 0$. Then if $\sigma_{11} - \sigma_{12} < 0$, $\sigma_{22} - \sigma_{12} > 0$, which implies the result for the mean of Sector 2 skills.

4. If $\sigma_{11} = \sigma_{12}$, there is *no selection bias* in Sector 1, i.e., mean of $\ln T_1$ employed in sector 1 *equals* μ_1. Furthermore, note that it is also the case that the variance of $\ln T_1$ employed in Sector 1 is equal to the variance of $\ln T_1$ in the population. Finally, note that if $\sigma_{11} = \sigma_{12} = \sigma_{22}$, there is no selection bias in either sector. In this case, the sorting across sectors would look as if agents were *randomly assigned* to the two sectors.

More on Effect of Self-Selection on Distribution of Earnings across Sectors

To gain further insight into the effect of self-selection on the distribution of earnings, consider the following:

Recall that under normality, the regression equation for $\ln T_2$ condition on $\ln T_1$ is given by:

$$\ln T_2 = \mu_2 + \frac{\sigma_{12}}{\sigma_{11}}\left(\ln T_1 - \mu_1\right) + \varepsilon_2, \tag{13}$$

where $E(\varepsilon_2) = 0$ and $\text{var}(\varepsilon_2) = \sigma_{22}\left[1 - \left(\sigma_{12}^2/\sigma_{11}\sigma_{22}\right)\right]$.

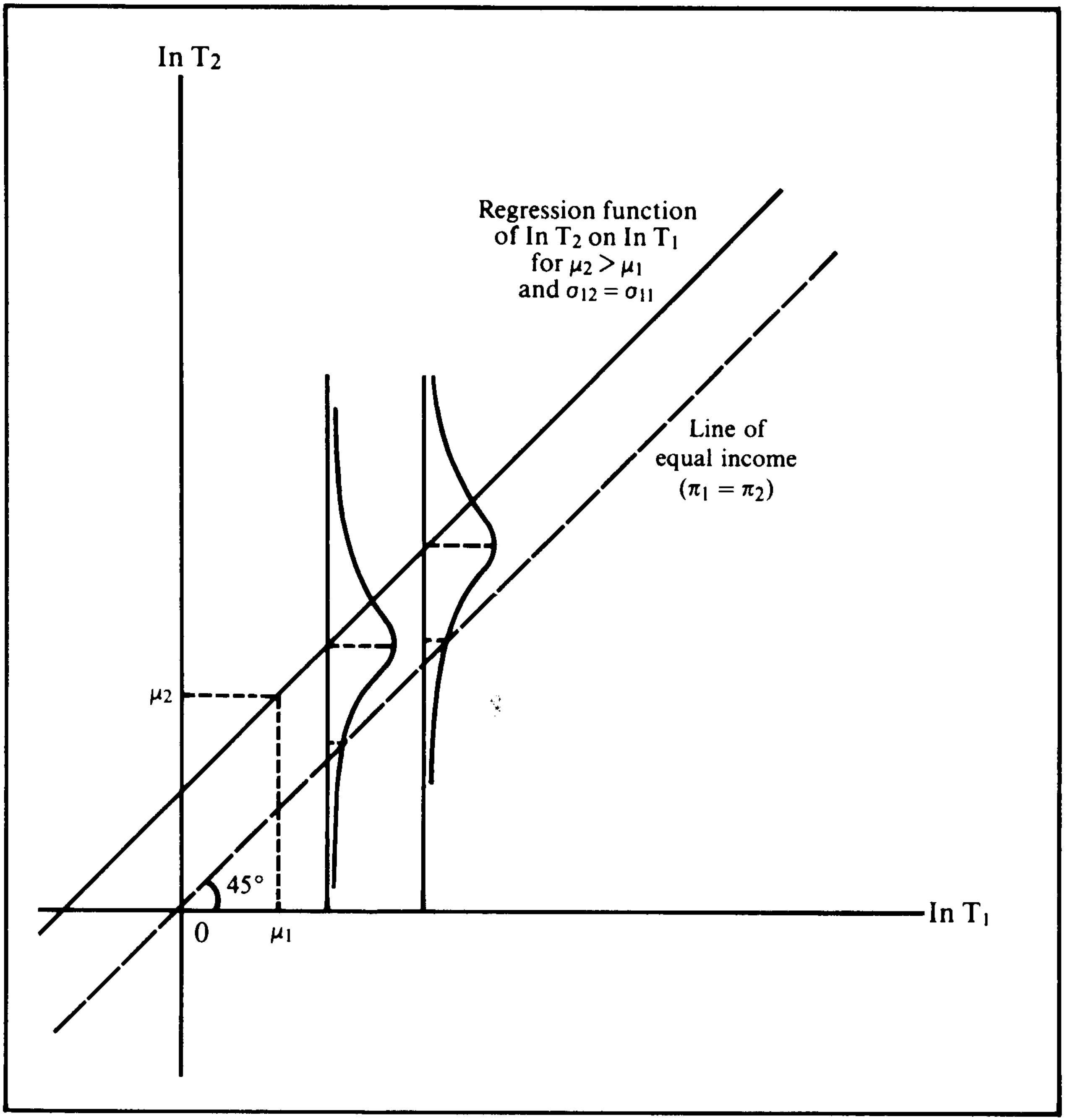
Consider **Figure 1** which plots (13) when $\sigma_{12} = \sigma_{11}$ [so $\sigma_{12} > 0$] and $\mu_2 > \mu_1 > 0$. In this case, agents with *high* values of $\ln T_1$ also tend to have *high* values of $\ln T_2$.

Points to note:

(a) If $\pi_1 = \pi_2$, agents with endowments of ($\ln T_1$, $\ln T_2$) above the 45° line (equal income line) choose to work in Sector 2 and those below choose to work in Sector 1.

(b) For any given value of $\ln T_1 = \ln t_k$, the same proportion of agents work in Sector 1, for all k. Therefore, the distribution of $\ln T_1$ employed in Sector 1 is the same as in the latent population distribution, i.e., self-selection in this case does not distort the population distribution of skills.

(c) If raise π_1 (or lower π_2), which shifts the 45° line upward, more agents now work in Sector 1 than before. But, it follows from (b) that the same proportion of people enter Sector 1 at each value of $T_1 = t_k$ for all k.



ln T2
Regression function
of ln T2 on ln T1
for μ2 > μ1
and σ12 = σ11
Line of
equal income
(π1 = π2)
μ2
45°
0
μ1
ln T1

Figure 1

Now consider **Figure 2** which plots (13) when $\sigma_{12} > \sigma_{11}$ [so still the case that $\sigma_{12} > 0$] and $\mu_2 > \mu_1 > 0$. Assume initially that $\pi_1 = \pi_2$.

Points to note:

(a) As we have already seen for this case, the mean of skill level in Sector is *lower* than the population mean level of T_1.

(b) Moreover, agents with high amounts of T_1 are under-represented in Sector 1. Why? Given $\pi_1 = \pi_2$, this occurs because $\mu_2 > \mu_1$, i.e., the typical agent will have a higher level of $\ln T_2$ than $\ln T_1$.

(c) Note that in the extreme case, where $\ln T_1$ and $\ln T_2$ are *perfectly positively correlated*, we have the extreme version of *absolute advantage* or *hierarchical sorting*. In this case, the highest paid worker in Sector 1 earns the same as the lowest paid worker in Sector 2! There is really only one skill and agents can be ranked by this skill.

(d) Now if we raise π_1 (or lower π_2), attracting workers to Sector 1, the mean of $\ln T_1$ must go up. For this to happen, workers from the *upper end* of the $\ln T_1$ distribution will switch to Sector 1. Furthermore, note that an x% increase in π_1 leads to a *more than* x% increase average $\ln W_1$ in Sector 1 since the average quality of workers in Sector 1 rose.

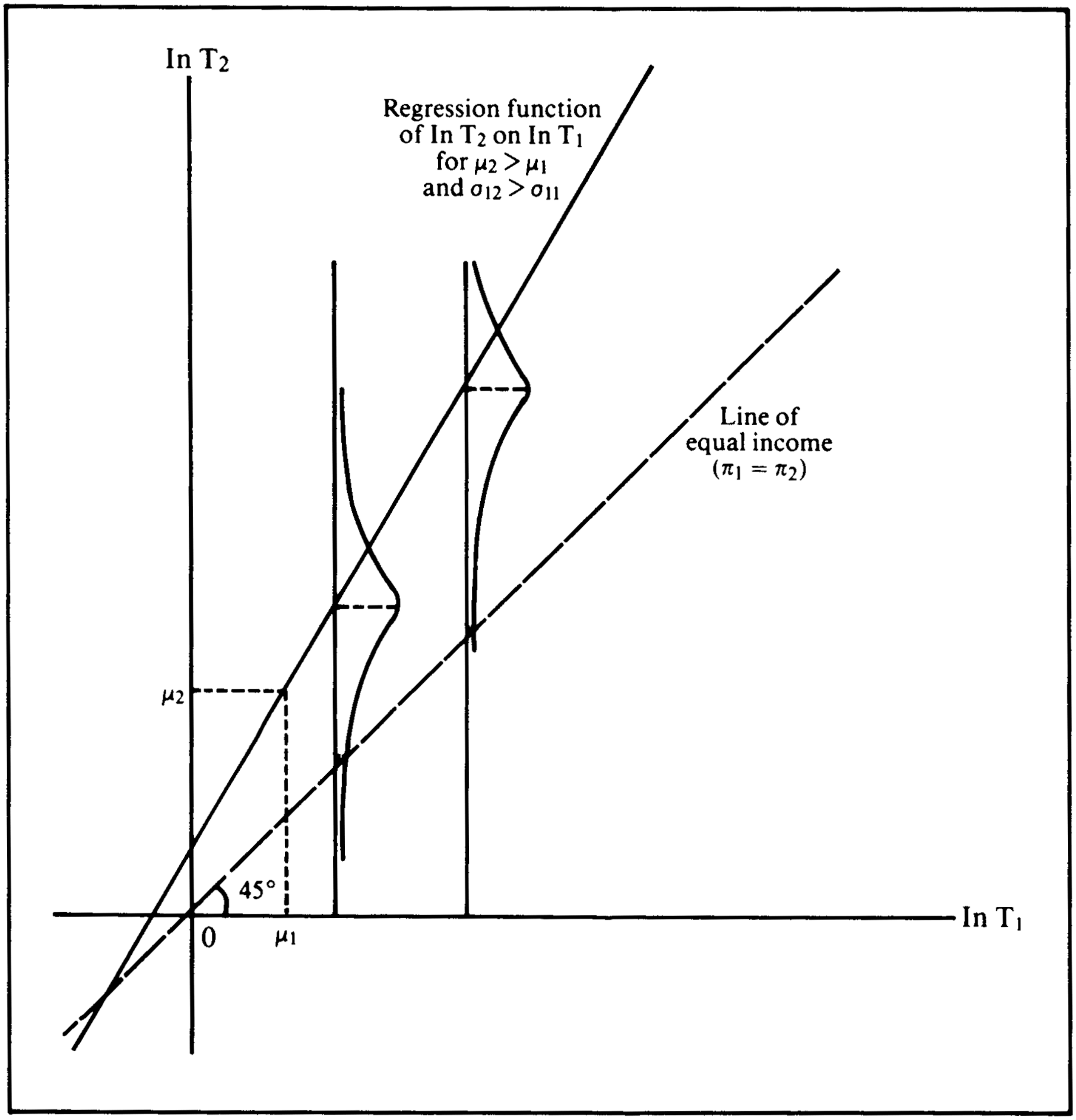
Finally, if we consider case where $\sigma_{12} < \sigma_{11}$ and $\mu_2 > \mu_1 > 0$, then

(a) Again, as we have already seen, mean of $\ln T_1$ will exceed μ_1 in equilibrium.

(b) Moreover, the proportion of workers from each $\ln T_1 = \ln t_k$ group working in this sector will *increase* with higher values of $\ln T_1$.

(c) Here, an x% *increase* in π_1 leads to a *less than* x% increase average wages ($\ln W_1$) in Sector 1 as the mean level of skills ($\ln T_1$) employed in Sector 1 declines.

(d) Note that it is possible that if $\sigma_{12} > \sigma_{22}$ an increase in π_1 can cause measured sector 1 wages to *decline*. Note that this can never happen if $\sigma_{12} < 0$ or, more generally *comparative holds*.



ln T2
Regression function
of ln T2 on ln T1
for μ2 > μ1
and σ12 > σ11
Line of
equal income
(π1 = π2)
μ2
45°
0
μ1
ln T1

Figure 2

The Empirical Content of the Roy Model with Normality

Heckman and his co-authors establish that a number of the above propositions do not hold if the normality assumption is relaxed. In particular:

(a) Increasing selection (as would result from changes in the π's) need not decrease sectoral variances (see Heckman & Sedlacek).

(b) The effect of selection on mean employed skill levels is also ambiguous (Heckman & Sedlacek and Heckman & Honoré). Therefore, predictions about what happens to mean skill levels in the U.S. as the returns to labor change are no longer readily predicted. This is because the simple properties of how truncated means change with skill prices no longer hold.

(c) Moreover, Heckman and Honoré establish that identification of the parameters of the Roy Model with observable data do not hold in a single cross-section of data without the normality assumptions used above. I'll focus on this last point.

Empirical Content of Roy Model?

Objective: We want to retrieve (identify) the joint distribution of $\ln T_1$ and $\ln T_2$

$$f(\ln T_1, \ln T_2) \tag{14}$$

from data on *observed* conditional distributions of wages,

$$\begin{gathered} g(\ln W_1 | \ln W_1 > \ln W_2) \\ g(\ln W_2 | \ln W_2 > \ln W_1) \end{gathered}, \tag{15}$$

and the distribution of sectoral choices, i.e.,

$$\Pr(i) = P(\ln W_i > \ln W_j), i \neq j, i, j = 1, 2. \tag{16}$$

The identification question is: When can we go from (15) and (16) to get (14)?

Answer:

1. If (T_1,T_2) are log normal, Heckman & Honoré prove that one can retrieve the parameters of the joint distribution in (14) from data on (15) and (16). Note that for the log normal case, the parameters, μ_1, μ_2, σ_{11}, σ_{22} and σ_{12} fully characterize the joint distribution in (14). Note that this result doesn't even require that we have data in which the prices (π's) *vary*!

2. If one doesn't assume the log normal distribution for (T_1,T_2) – and more importantly doesn't know the form of the distribution – ***and*** one only has a ***single cross-section*** of data – i.e., there is ***no price variation*** – Heckman & Honoré prove that one ***cannot*** retrieve (14) from data on (15) and (16). That is, they prove a non-identifiability result.

3. Suppose we don't know the distribution of (T_1,T_2) but we do have data for ***repeated cross-sections*** – could be panel data – where the important feature here is that we ***do have exogenous variation in prices***. Then Heckman & Honoré prove that one can retrieve (14) from data on (15) and (16) under the following assumptions:

   (i) Agents are pure income maximizers, i.e., other factors don't enter into their utility or payoffs from being in a particular sector.

   (ii) There is, in principle, variation in prices across the full range of prices or that make some sort of "continuity" assumption concerning the effect of prices on wages.

   (iii) $f(\cdot,\cdot)$ is stable across time or economies, depending on where variation in prices comes from.

   (iv) Prices are known to the econometrician.

   Line of argument in proof is as follows:

   Focus on variation in relative prices by normalizing $\pi_1 = 1$. Suppose we have data on wages, W, and we have variation in skill prices across economies. (Note, we don't actually need to know which sector agent chose.) Let π_2 vary from $(0,\infty)$. It follows that:

$$\begin{aligned}\Pr\left(W \le n\right) &= \Pr\left(\max\left(T_1,\pi_2 T_2\right) \le n\right) \\ &= \Pr\left(T_1 \le n, T_2 \le \frac{1}{\pi_2}n\right) \\ &= F\left(n,\frac{n}{\pi_2}\right)\end{aligned} \tag{17}$$

   So as π_2 varies between 0 and ∞, we can trace out the entire distribution, $F(\cdot,\cdot)$. Can also impose some restrictions imposed on form of $F(\cdot,\cdot)$, we don't need as much variation in π_2.

4. If we have panel data, then we exploit to identify $F(\cdot,\cdot)$ with just variation in prices for two different periods, so long as we *assume* that each *individual's skills don't change over time* (which is more plausible if we have panel data on the same individuals).

   We observe wages for agent i, i = 1,.., N, in two periods, t and t', i.e., we observe W_{it} and $W_{it'}$, for which we know

$$\left(W_{it}, W_{it'}\right) = \left(\max\left\{T_1, \pi_2 T_2\right\}, \max\left\{T_1, \pi_2' T_2\right\}\right) \tag{18}$$

   where we observe prices, π_2, π_2', where $\pi_2 < \pi_2'$. (Reversing this inequality makes no difference to the logic of the proof.)

   Consider values, t_1, $t_2 > 0$ such that $\pi_2 t_2 \le t_1 \le \pi_2' t_2$. Then

$$\begin{aligned} F(t_1,t_2) &= \Pr\left(T_1 \le t_1, T_2 \le t_2\right) \\ &= \Pr\left(T_1 \le t_1, T_1 \le \pi_2' t_2, T_2 \le t_2, T_2 \le t_1/\pi_2\right) \\ &= \Pr\left(T_1 \le t_1, \pi_2 T_2 \le t_1, T_1 \le \pi_2' t_2, \pi_2' T_2 \le \pi_2' t_2\right) \\ &= \Pr\left(\max\left\{T_1, \pi_2 T_2\right\} \le t_1, \max\left\{T_1, \pi_2' T_2\right\} \le \pi_2' t_2\right) \\ &= \Pr\left(W_{it} \le t_1, W_{it} \le \pi_2' t_2\right) \end{aligned} \tag{19}$$

   where the last step follows from our knowledge of the distribution of observed wages for each agent i. Thus, all we needed was variation in prices, i.e., π_2, π_2', $\pi_2 < \pi_2'$.

5. Heckman & Honoré also show that allowing the μ's to be functions of observables, x, allows us to identify $F(\cdot,\cdot)$ *without* having variation in prices. See Theorem 12 in their paper.